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ON THE COMPLETENESS OF STURM-LIOUVILLE EXPANSIONS

GARRETT BIRKHOFF, Harvard University, AND GIAN-CARLO ROTA,* Massachusetts Institute of Technology

1. Historical remarks. A very old difficulty in the theory of ordinary differential equations is the proof of the fundamental assertion about the eigenfunctions of a Sturm-Liouville system: that such a sequence of functions is complete. In the special case of $y''+\lambda y=0$, this difficulty can be resolved by elementary methods taught in advanced calculus (See, for example, [4], Ch. IX or [10], Ch. XIII). However, it should be remembered that, even in this case, the difficulty perplexed such mathematical giants as Euler and Lagrange, being finally resolved rigorously by Dirichlet and Riemann† only after a century of controversy.

The general case is correspondingly more difficult, though the difficulty is commonly glossed over by physicists. In his celebrated *Theory of Sound* (Art. 187), Rayleigh discussed the number of zeros of the remainder of an "arbitrary function" after the first n terms of the Sturm-Liouville series have been subtracted from it. He then asserted that "the application of this theorem to demonstrate the possibility of expanding an arbitrary function in an infinite series of normal functions would proceed exactly as in Art. 142," that is, as in the case of Fourier series. Sommerfeld (*Partial Differential Equations*, p. 179) refers to this result as the *Ohm-Rayleigh Principle.*

Apparently Bôcher queried Lord Rayleigh about his statement. His question received the following reply, in 1899; a photostat of it is reproduced in Figure 1 because of its historical interest.

> Dear Prof. Bôcher
>
> I doubt if I had any special method in my mind when I wrote §187. At any rate if I had then, I have not now.
>
> I think I took this case as covered by the general theory as in Ch. IV. I have often wished that mathematicians would devote themselves to improving the rigour of this and similar demonstrations, rather than to inventing special methods for the particular cases.
>
> Prof. Klein, who was here a few months ago, spoke of some recent work of Poincaré in this direction. Probably you know of this. I think Klein specially mentioned something published at Palermo.

Actually, Rayleigh could have justified his statement quite easily, had he combined already available results with some simple observations about Hilbert space. It is the purpose of this note to supply the needed argument; it follows lines originally suggested by George D. Birkhoff [1].

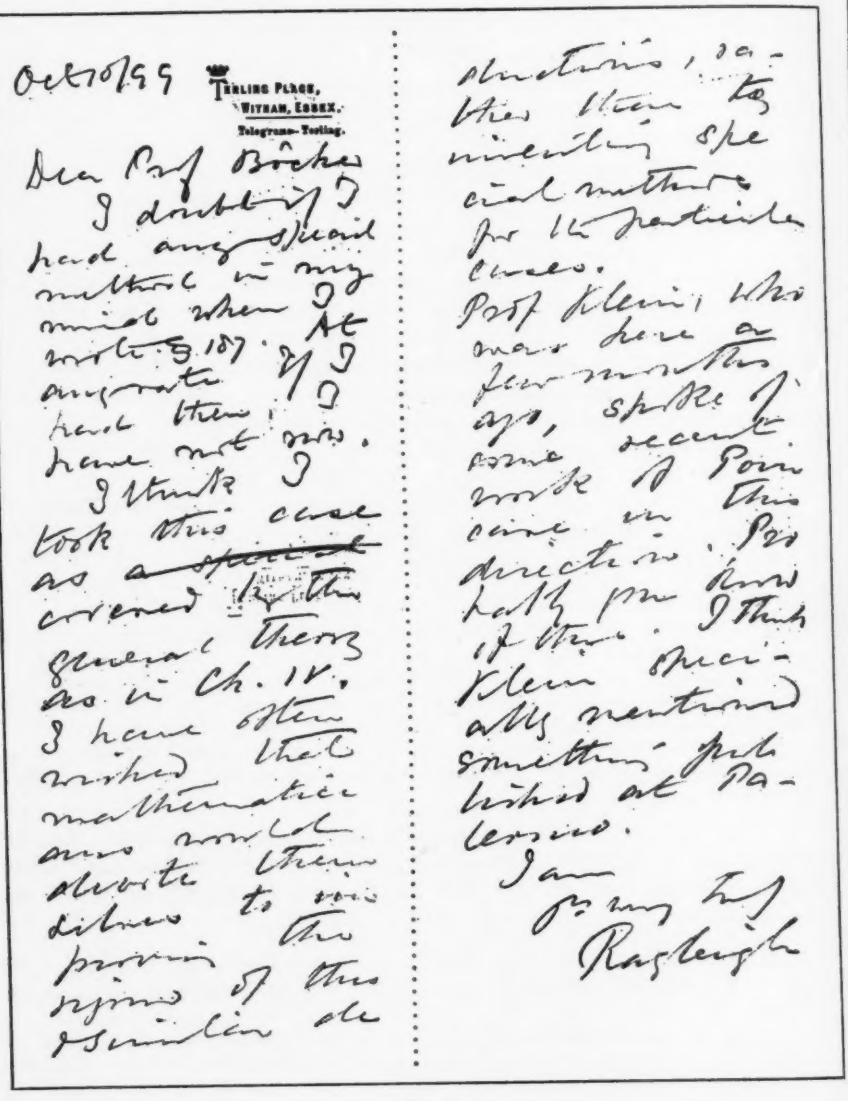
2. Role of Hilbert space. Let $\mathfrak{C}$ be the Euclidean vector space ([1], p. 189) of all real functions $y=f(x)$ continuous on $a\leqq x\leqq b$. Given $f\in\mathfrak{C}$ and a sequence

* Professor Rota's work was partially done under Contract 7667, Office of Naval Research.

† See [6]. Fourier's arguments were hardly rigorous; he referred loosely to "all possible functions" (Analytical Theory of Heat, end of Art. 224).

$\{\phi_k\}$ of normalized orthogonal functions, one can define the formal expansion

$$f \sim c_1\phi_1 + c_2\phi_2 + \cdots, \qquad c_k = (f, \phi_k) = \int_a^b f\phi_k dx. \tag{1}$$



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Dear Prof Bôcher

I doubt if I had any special method in my mind when I wrote §. 187. At any rate I had then I have not now. I think I took this case as a special case covered by the general theory as in Ch. IV. I have often wished that mathematicians would devote their labours to the proving of the reality of similar de-

ductions, as this item to investigating special methods for the particular cases. Prof Klein, who was here a few months ago, spoke of some recent work of Poincaré in this connection. Perhaps you know of this. I think Klein specially mentioned something published at Palermo.

I am

Yours truly

Rayleigh

FIG. 1

The coefficients c_k always satisfy the Bessel inequality

$$\sum_1^\infty c_k^2 \leqq \int_a^b f^2(x)dx < +\infty. \tag{2}$$

The sequence $\{\phi_k\}$ is called *complete* when, for every $f \in \mathfrak{C}$, the Bessel inequality (2) is an equality (the Parseval equality):

$$\sum_1^\infty c_k^2 = \sum_1^\infty (f, \phi_k)^2 = \int_a^b f^2(x)dx. \tag{2*}$$

It is proved in advanced calculus that the cosine functions

$$\phi_k = \sqrt{(2/\pi)} \cos kx \qquad (k = 0, 1, \cdots)$$

are complete on $0 \leqq x \leqq \pi$, as are the sine functions

$$\bar{\phi}_k = \sqrt{(2/\pi)} \sin kx \qquad (k = 1, 2, \cdots);$$

these are correlated with the even and odd components $\frac{1}{2}[f(x)+f(-x)]$ and $\frac{1}{2}[f(x)-f(-x)]$, respectively, of a general continuous function on $-\pi \leqq x \leqq \pi$.

The expansion (1) establishes a linear isomorphism $\theta: f \to (c_1, c_2, \cdots)$ between $\mathfrak{C}$ and a subspace $\theta(\mathfrak{C}) = \mathfrak{G}$ of the space $\mathfrak{H}$ of all infinite sequences $(c_1, c_2, \cdots)$ satisfying $\sum_1^\infty c_k^2 < +\infty$. Moreover, if (real) inner products are defined in $\mathfrak{H}$ by

$$(\gamma, \tilde{\gamma}) = \sum_1^\infty c_k \tilde{c}_k, \qquad \gamma = (c_1, c_2, \cdots), \qquad \tilde{\gamma} = (\tilde{c}_1, \tilde{c}_2, \cdots), \tag{3}$$

then the mapping θ preserves inner products: if $\theta(f) = \gamma$ and $\theta(\tilde{f}) = \tilde{\gamma}$, then $\int_a^b f(x)\tilde{f}(x)dx = (\gamma, \tilde{\gamma})$. The definition (3) makes $\mathfrak{H}$ a Euclidean vector space, and in fact a *Hilbert space* l_2; relative to the usual distance formula

$$\|\gamma - \tilde{\gamma}\| = (\gamma - \tilde{\gamma}, \gamma - \tilde{\gamma})^{1/2} \tag{3'}$$

defined in $\mathfrak{H}$ by (3), θ is distance-preserving (an *isometry*).

Now consider a general regular Sturm-Liouville system. It suffices to consider the differential equation

$$y'' + (\lambda - q(x))y = 0, \qquad 0 \leqq x \leqq \pi, \tag{4}$$

since other cases can be reduced to this by Liouville's transformation ([9], p. 238; [5], pp. 291–295.) We assume *separated* endpoint conditions, of the form

$$\alpha f(0) + \alpha_1 f'(0) = \beta f(\pi) + \beta_1 f'(\pi) = 0. \tag{4'}$$

As essentially shown by Liouville, one can then derive the following asymptotic estimate ([9], p. 244; [3], Part B) for the eigenfunctions of (4)–(4'):

$$\psi_k(x) = \sqrt{(2/\pi)} \cos kx + O(1/k), \tag{5}$$

provided $\alpha_1 \neq 0$ and $\beta_1 \neq 0$. (If $\alpha_1 = 0$ or $\beta_1 = 0$, use sine in place of cosine.) As a

corollary, we have the relation $\|\psi_k-\phi_k\|^2=O(1/k^2)$, where ϕ_k is an orthonormal basis* of cosine or sine functions. Hence, in particular,

$$\sum_{k=1}^{\infty}\|\psi_k-\phi_k\|^2<+\infty. \tag{5'}$$

3. Main theorem. The completeness of the ψ_k now follows from a general theorem about bases in Hilbert space. This can be made to follow from more general theorems on biorthogonal systems ([8], p. 206), but we have not seen a simple direct proof in the literature.

THEOREM 1. *Let ϕ_k $(k=0, 1, \cdots)$ be an orthonormal basis in Hilbert space $\mathfrak{H}$ and let ψ_k $(k=0, 1, \cdots)$ be an orthonormal sequence in $\mathfrak{H}$ satisfying*

$$\sum_{k=1}^{\infty}\|\psi_k-\phi_k\|^2<+\infty. \tag{6}$$

Then the sequence $\{\psi_k\}$ is also an orthonormal basis.

LEMMA. *Let $\phi_0, \phi_1, \cdots$ be an orthonormal basis in the Hilbert space $\mathfrak{H}$. Let $\psi_0, \psi_1, \cdots$ be an orthonormal sequence in $\mathfrak{H}$ satisfying the conditions*

$$\sum_{k=1}^{\infty}\|\psi_k-\phi_k\|^2<1. \tag{7}$$

Then the sequence $\{\psi_k\}$ is also an orthonormal basis in $\mathfrak{H}$.

Proof. If the sequence $\{\psi_k\}$ were not a basis, then we could find a nonzero function g orthogonal to every ψ_k. For this function, $(g, \phi_k)=(g, \psi_k)+(g, \phi_k-\psi_k)=(g, \phi_k-\psi_k)$. Squaring and using the Schwarz inequality,

$$(g, \phi_k)^2=(g, \phi_k-\psi_k)^2\leqq\|g\|^2\|\phi_k-\psi_k\|^2. \tag{7'}$$

Summing with respect to k, we get

$$\sum_{k=0}^{\infty}(g, \phi_k)^2\leqq\|g\|^2\sum_{k=0}^{\infty}\|\phi_k-\psi_k\|^2<\|g\|^2,$$

the $\{\phi_k\}$ being a basis, in evident violation of Parseval's equality.

COROLLARY 1. *Replace condition* (7) *above by the weaker condition*

$$\sum_{k=N+1}^{\infty}\|\psi_k-\phi_k\|^2<1$$

for some integer N. Then every element of $\mathfrak{H}$ orthogonal to $\phi_0, \phi_1, \cdots, \phi_N$ and to $\psi_{N+1}, \psi_{N+2}, \cdots$ must vanish.

* By a *basis* for the Hilbert space $\mathfrak{H}$, we mean a subset, whose finite linear combinations are dense in $\mathfrak{H}$.

Proof. Any such element g would satisfy the inequality (7′) for all $k>N$. Hence, summing over all k, since $(g, \phi_k)=0$ for $k=1, \cdots, N$, we have

$$\sum_{k=1}^{\infty}(g, \phi_k)^2 = \sum_{k=N+1}^{\infty}(g, \phi_k)^2 \leqq \|g\|^2 \sum_{k=N+1}^{\infty}\|\phi_k - \psi_k\|^2 < 1,$$

again contradicting Parseval's equality (2*).

COROLLARY 2. *Under the assumptions of Corollary* 1, *every element of* $\mathfrak{H}$ *orthogonal to* $\psi_{N+1}, \psi_{N+2}, \cdots$ *and to the elements*

$$\eta_n = \phi_n - \sum_{k=N+1}^{\infty}(\phi_n, \psi_k)\psi_k, \qquad n = 0, 1, \cdots, N, \tag{8}$$

must vanish.

Proof. For any such element g we have

$$(g, \phi_n) = (g, \eta_n) + \sum_{k=N+1}^{\infty}(\phi_n, \psi_k)(g, \psi_k) = 0; \quad n = 0, 1, \cdots, N.$$

Thus g also satisfies the conditions of Corollary 1, and therefore vanishes.

The proof of Theorem 1 can now be completed as follows.

Choose an integer N so that

$$\sum_{k=N+1}^{\infty}\|\psi_k - \phi_k\|^2 < 1.$$

By Corollary 2, any element g of $\mathfrak{H}$ which is orthogonal to the elements ψ_{N+1}, $\psi_{N+2}, \cdots$ and to the elements η_n $(n=0, 1, \cdots, N)$ defined by (8) must vanish. Denote by S the set of all elements of $\mathfrak{H}$ orthogonal to $\psi_{N+1}, \psi_{N+2}, \cdots$. Evidently S is a vector space containing $\eta_0, \eta_1, \cdots, \eta_n$. By virtue of the above remark, the vector space S contains only the linear combinations of these elements. In other words, S is a finite-dimensional vector space whose dimension is at most N.

But the elements $\psi_1, \cdots, \psi_N$ also belong to the vector space S, and they are linearly independent (they are an orthonormal sequence!). Therefore the elements $\psi_1, \cdots, \psi_N$ are a *basis* for the vector space S ([2], pp. 168–9). Hence the elements $\eta_0, \eta_1, \cdots, \eta_N$ are linear combinations of $\psi_1, \cdots, \psi_N$. It follows that any element of $\mathfrak{H}$ which is orthogonal to all the ψ_k must vanish, because such an element is also orthogonal to $\eta_0, \eta_1, \cdots, \eta_N$ and to $\psi_{N+1}, \psi_{N+2}, \cdots$. We conclude that the sequence $\{\psi_k\}$ is complete.

4. Further discussion. As Rayleigh, like most of his contemporaries, probably interpreted "completeness" as referring to pointwise or uniform convergence, it is relevant to discuss the relation between the "mean-square convergence" defined by the Parseval equality, and pointwise or uniform con-

vergence. Some facts can be proved easily, without an appeal to the Lebesgue integral.

Thus, from the Weierstrass approximation theorem (surely known to Rayleigh!), it follows immediately that the set $\mathfrak{P}$ of *polynomial* functions is dense in $\mathfrak{H}$. Therefore, the functions of class $\mathfrak{C}'$, $\mathfrak{C}''$, etc. (the number of primes referring to continuous differentiability) are also dense in $\mathfrak{H}$. By a slight modification near the endpoints, one can however approximate any such function by one satisfying the separated endpoint conditions (4'), arbitrarily closely in $\mathfrak{H}$. We now observe

THEOREM 2. *If $f \in \mathfrak{C}''$ satisfies the endpointcon ditions* (4'), *then the kth term of the expansion of f in the eigenfunctions ψ_k of the Sturm-Liouville system* (4)–(4') *is $O(1/k^2)$.*

Proof. Equation (4) implies

$$c_k = \int_0^\pi f(x)\psi_k(x)dx = (1/\lambda_k)\int_0^\pi [q(x)\psi_k(x) - \psi_k''(x)]f(x)dx. \tag{9}$$

On the other hand, by the Lagrange identity and (4')

$$\int_0^\pi (\psi_k f'' - f\psi_k'')dx = [\psi_k f' - f\psi_k']_{x=0}^{x=\pi} = 0,$$

since f and the ψ_k both satisfy (4'). Hence (9) implies

$$c_k = (1/\lambda_k)\int_0^\pi [q(x)\psi_k(x) - \psi_k(x)f''(x)]dx = O(1/k^2), \tag{9*}$$

where the last equality follows since $\lambda_k = O(k^2)$, while the ψ_k, q, and f'' are uniformly bounded, all by (5). Uniform convergence follows from (9*) by the Weierstrass M-test.

So far, Lebesgue integration has not been invoked. This is needed to identify $\mathfrak{H}$ as constructed above with the class $L_2(0, \pi)$ of functions whose squares are Lebesgue integrals. Specifically, given any formal sum $\sum_1^\infty c_k\phi_k(x)$ with $\sum_1^\infty c_k^2 < +\infty$, one can extract from the partial sums $\sigma_n(x) = \sum_1^n c_k\phi_k(x)$ a subsequence $\sigma_{n(k)}(x)$ converging almost everywhere to a (Lebesgue integrable) limit $\sigma_\infty(x)$; this is the essence of the proof of the Riesz-Fischer Theorem ([7], p. 185); any two such limits are equal a.e., and determine the same σ_∞ in $\mathfrak{H}$. Using these facts, one can prove

THEOREM 4. *If the nth partial sums $\sigma_n(x) = \sum_1^n c_k\phi_k(x)$ in* (1), *converge pointwise to $f(x)$ then they converge in the mean square.*

For, by hypothesis, the $\sigma_{n(k)}(x)$ mentioned above converge pointwise to $f(x)$; hence $f(x)$ coincides with $\sigma_\infty(x)$ almost everywhere, implying the Parseval equality (2*). We omit the details.

COROLLARY. *If the nth partial sums $\sigma_n(x)$ in (1) converge pointwise to $f(x)$, for all f in any dense subset $\mathfrak{D}$ of $\mathfrak{H}$, then $\{\phi_k\}$ is complete.*

It is not clear that the above results can be proved without using the Lebesgue integral. They provide a final link between classical pointwise convergence and the modern theory of mean square convergence.

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OPERATORS AND THE r-WAY CROSSED CLASSIFICATION

J. D. BANKIER, McMaster University

1. Introduction. The development of the analysis of variance for the r-way crossed classification usually involves some tedious algebra. It is our purpose to clear much of this algebra away by means of an operational approach to the problem. The resulting presentation has been used successfully with undergraduates.

2. The 2-way crossed classification. It will be easier to understand what follows if we first review some of the results for the case where $r=2$. Here we make observations, X_{ijk}, and are concerned about the effect upon these observations of two factors, temperature and pressure, say. We make a selection of p temperatures and q pressures and make n observations at each of the pq possible combinations of temperature and pressure. We assume that the expected value of X_{ijk} is given by the equation

$$(1)\quad E(X_{ijk}) = \mu + \alpha_i + \beta_i + (\alpha\beta)_{ij} \quad (i = 1, \cdots, p; j = 1, \cdots, q; k = 1, \cdots, n),$$

where μ is a constant, α_i and β_j are the *main effects* of the ith temperature and jth pressure while $(\alpha\beta)_{ij}$ is called the *interaction* between the ith temperature and the jth pressure. As we shall see in the general case, we may assume without loss of generality that

$$\sum_{i=1}^{p} \alpha_i = \sum_{j=1}^{q} \beta_j = \sum_{i=1}^{p} (\alpha\beta)_{ij} = \sum_{j=1}^{q} (\alpha\beta)_{ij} = 0. \tag{2}$$

We are interested in testing three hypotheses; the first is that the temperatures all have the same effect and hence, by (2), all α_i are equal to zero. The remaining hypotheses are $\beta_j = 0$ and $(\alpha\beta)_{ij} = 0$ for all i and j. To test the hypotheses we form an analysis of variance table containing the quadratic forms

$$SSA = nq \sum_{i=1}^{p} (\overline{X}_{i\cdot\cdot} - \overline{X}_{\cdot\cdot\cdot})^2, \qquad SSB = np \sum_{j=1}^{q} (\overline{X}_{\cdot j\cdot} - \overline{X}_{\cdot\cdot\cdot})^2,$$

$$SS(AB) = n \sum_{i,j} (\overline{X}_{ij\cdot} - \overline{X}_{i\cdot\cdot} - \overline{X}_{\cdot j\cdot} + \overline{X}_{\cdot\cdot\cdot})^2, \qquad SSE = \sum_{i,j,k} (X_{ijk} - \overline{X}_{ij\cdot})^2,$$

$$SST = \sum_{i,j,k} (X_{ijk} - \overline{X}_{\cdot\cdot\cdot})^2,$$

where subscripts replaced by dots have been averaged over. The above quadratic forms satisfy the identity $SST = SSA + SSB + SS(AB) + SSE$ which may be rewritten as

$$\sum_{i,j,k} X_{ijk}^2 = npq\overline{X}_{\cdot\cdot\cdot}^2 + SSA + SSB + SS(AB) + SSE. \tag{3}$$

The ranks, or degrees of freedom, of the quadratic forms are $p-1$, $q-1$, $(p-1)(q-1)$, $pq(n-1)$ and $npq-1$, respectively. The mean sums of squares, MSA, MSB, $MS(AB)$ and MSE are obtained by dividing the corresponding sums of squares by their degrees of freedom and it is the ratio of the first three mean sums of squares to the fourth which are used to test the three hypotheses. More detail and proofs will be given in the succeeding sections of this paper when we consider the general case, our approach being one which was used by Wald [2].

3. The r-way crossed classification. The reader may find it helpful to set $r = 2$ in the work that follows and compare the results with those given in Section 2. An observation is denoted by $X_a = X_{a_1 \cdots a_{r+1}}$, $a_i = 1, \cdots, t_i$, where the last subscript is used to indicate replications. Main effects and interactions are represented by $\mu(I, a_I) = \mu(i_1, \cdots, i_k; a_{i_1}, \cdots, a_{i_k})$, where $I = (i_1, \cdots, i_k)$ is a subset of $R = (1, \cdots, r)$ and $\mu(I, a_I) = \mu$ if I is the null set.

We introduce two operators, $D_i = D_{a_i}$ which drops i and a_i from $\mu(R, a_R)$ and $M_i = M_{a_i}$ which averages a function over a_i, if a_i appears, and otherwise leaves the function unchanged. These operators obey all the ordinary laws of algebra, excluding division. Corresponding to equations (1) and (2), we assume that

$$E(X_a) = (1 + D)_R \mu(R, a_R), \tag{4}$$

$$M_j \mu(I, a_I) = 0, \qquad (j = 1, \cdots, k), \tag{5}$$

where $(1+D)_R = (1+D_1) \cdots (1+D_r)$.

It is easy to establish

LEMMA 1. *Independent of condition* (5),

(6) $M_iD_i=D_i,\ (1-M_i)D_i=0,\ M_i(1+D_i)=M_i+D_i,\ (1-M_i)(1+D_i)=1-M_i.$

Subject to condition (5),

(7) $(M_i+D_i)\mu(R, a_R)=D_i\mu(R, a_R),\ (1-M_i)\mu(R, a_R)=\mu(R, a_R),$

provided that these expressions are not multiplied by D_i.

We define the sums of squares

(8) $$SS(I)=tM_IA^2(I, a_I),$$

where $t=t_1\cdots t_{r+1}$, I is a subset, not necessarily proper, of the set $R+1=(1, \cdots, r+1)$, $M_I=M_{i_1}\cdots M_{i_k}$,

(9) $$A(I, a_I)=M_{R+1-I}(1-M)_IX_a,$$

$M_{R+1-I}=M_1\cdots M_{r+1}/M_{i_1}\cdots M_{i_k}$ and $(1-M)_I=(1-M_{i_1})\cdots(1-M_{i_k})$. We note that, if I is the null set, $A=A(0, a_0)=M_{R+1}X_a=\overline{X}$, where $\overline{X}$ is the mean of all observations and $SS(0)=t\overline{X}^2$. We also define an *error sum of squares*, SSE, as the sum of all sums of squares such that $r+1$ is a member of the set I so that

(10) $$SSE=(1+D)_RSS(R+1).$$

We now establish the fundamental result

LEMMA 2. $A(I, a_I)=M_b(t\delta_{ab}-1)_IX_b$, *where* $M_b=M_{b_1}\cdots M_{b_{r+1}}$.

Proof. Note that $(1-M_1)X_a=M_{b_1}(t_1\delta_{a_1b_1}-1)X_{b_1a_2\cdots a_{r+1}}$. Then the desired result follows from definition (9).

We are now in a position to establish a result corresponding to (3). It is

THEOREM 1. $tMX_a^2=(1+D)SS(R+1)$, *where* $t=t_1\cdots t_{r+1}$, $M=M_{R+1}=M_1\cdots M_{r+1}$, *and* $(1+D)=(1+D)_{R+1}=(1+D_1)\cdots(1+D_{r+1})$.

Proof. Making use of equations (8), (9) and Lemma 2, we have

$$\begin{aligned}(1+D)SS(R+1)&=t(1+D)_{a_{R+1}}M_aA^2(R+1, a_{R+1})\\&=t(1+D)_{a_{R+1}}M_aM_bM_c(t\delta_{ab}-1)_{R+1}(t\delta_{ac}-1)_{R+1}x_bx_c\\&=t(1+D)_{a_{R+1}}M_bM_c(t\delta_{bc}-1)_{R+1}x_bx_c\\&=tM_bM_c(1+t\delta_{bc}-1)_{R+1}x_bx_c=tM_bx_b^2.\end{aligned}$$

COROLLARY 1. *The analysis of variance table adds up since*

$$SST=SS(1)+\cdots+SS(r)+SS(1, 2)+\cdots+SS(r, r-1)+\cdots+SS(1, \cdots, r)+SSE,$$

where $SST=tM(X_a-\overline{X})^2$.

COROLLARY 2. $SS(I) = tM_bM_c(t\delta_{bc}-1)_IX_bX_c$.

We also note that

$$\sum_{a_{ij}=1}^{t_{ij}} A(I, a_I) = M_b \sum_{a_{ij}=1}^{t_{ij}} (t\delta_{ab} - 1)_I X_b = 0 \tag{11}$$

since

$$\sum_{a_{ij}=1}^{t_{ij}} (t_i \delta_{a_{ij}b_{ij}} - 1) = 0.$$

Mathematical induction and (11) may be used to verify that any $A(I, a_I)$, which has at least one of the $a_{i_j}=1$, can be expressed as a linear combination of terms such as $A(I, b_I)$ where no b_{i_j} is equal to unity. This result leads to a set of linearly independent linear homogeneous equations in the $A(I, a_I)$ and we may conclude that the rank of $SS(I)$, a quadratic form in the X_a, does not exceed $(t-1)_I$. Referring to Theorem 1, we see that

$$t = \text{rank } (tMX^2) \leqq (1 + D)(t - 1)_{R+1} = [1 + (t - 1)]_{R+1} = t \tag{12}$$

and this is only possible if the rank of $SS(I)$ is $(t-1)_I$.

To simplify (10), our definition of SSE, we apply the second corollary of Theorem 1 to obtain

$$\begin{aligned} SSE &= tM_bM_c[(1 + D)_R(t\delta_{bc} - 1)_{R+1}]X_bX_c \\ &= tM_bM_c(t\delta_{b_{r+1}c_{r+1}} - 1)t_R\delta_{bc_R}X_bX_c = tM(X_b - \overline{X}_{b_R})^2 \end{aligned}$$

where $\overline{X}_{b_R}$ is the mean obtained by averaging X_b over b_{r+1}. We observe that the degrees of freedom of SSE cannot exceed $(1+D)_R(t-1)_{R+1}=t_R(t_{r+1}-1)$, and an argument similar to that associated with (12) shows that $t_R(t_{r+1}-1)$ is indeed the rank of SSE. Equation (12) also shows that the degrees of freedom in our analysis of variance table add up to those of the total sum of squares, SST.

4. The r-way crossed classification without replication. All definitions and results of the preceding section hold with the understanding that $t_{r+1}=1$. We denote our observations by X_{a_R} and define

$$\mu_{a_R} = E(X_{a_R}) = (1 + D)_R\mu(R, a_R)$$

and

$$\begin{aligned} \bar{\mu}_{a_I} &= M_{R-I}\mu_{a_R} = M_{R-I}(1 + D)_R\mu(R, a_R) = (1 + D)_I(M + D)_{R-I}\mu(R, a_R) \\ &= (1 + D)_I D_{R-I}\mu(R, a_R) = (1 + D)_I\mu(I, a_I), \end{aligned}$$

making use of Lemma 1. Multiplying both sides of this result

$$\bar{\mu}_{a_I} = (1 + D)_I\mu(I, a_I) \tag{13}$$

by $(1-M)_I$ and applying Lemma 1 and equation (5) we find that

$$\mu(I, a_I) = (1 - M)_I \bar{\mu}_{a_I} = M_{R-I}(1 - M)_I \mu_{a_R}. \tag{14}$$

It is easy to verify that (14) does give us the unique solution of (13) and, if we make (14) our definition of $\mu(I, a_I)$, that our earlier definitions (4) and (5) are satisfied. Wald defined the main effects by $\mu(i_1; a_{i_1}) = \bar{\mu}_{a_{i_1}} - \bar{\mu}$, and the interactions by

$$\mu(I, a_I) = \bar{\mu}_{a_I} - [(1 + D)_I - 1]\mu(I, a_I) \tag{15}$$

where he subtracted interactions of lower order corresponding to subsets of a_I from the mean. This is a recursive way of obtaining (14) since (15) is merely (13) rewritten. Since the formulas for $A(I, a_I)$ and $\mu(I, a_I)$ are of the same form, $A(I, a_I)$ is an unbiased estimator of $\mu(I, a_I)$.

Up to this point, our work has been algebra rather than statistics; we now turn to the statistical aspects of the subject. We shall see that, when there is no replication, we must assume that all the $(r-1)$st order interactions, $\mu(R, a_R)$, are equal to zero. We introduce the hypotheses

$$H(I): \mu(I, a_I) = 0 \qquad (a_{i_j} = 1, \cdots, t_{i_j}; j = 1, \cdots, k).$$

We define $\epsilon_{a_R} = X_{a_R} - \mu_{a_R}$, and assume that the X_{a_R} are $NID(\mu_{a_R}; \sigma^2)$, that is, they are normally and independently distributed with means μ_{a_R} and a common variance σ^2. Also $Z_{a_R} = (X_{a_R} - \mu_{a_R})/\sigma$ and hence are $NID(0; 1)$. We define

$$A^*(I, a_I) = M_{R-I}(1 - M)_I Z_{a_R}, \qquad SS^*(I) = t_R M_I A^{*2}(I, a_I).$$

An application of Theorem 1 shows that $t_R M_R Z^2_{a_R} = (1+D)_R SS(R)$ and a consequence of Cochran's theorem ([3], p. 107) is

THEOREM 2. *The quadratic forms $SS^*(I)$, $0 \leqq k \leqq r$, are independently distributed as χ^2 with $(t-1)_I$ degrees of freedom (d.f.).*

It is easy to verify, with the help of (14), that

$$A(I, a_I) = \mu(I, a_I) + \sigma A^*(I, a_I) = \mu(I, a_I) + M_{R-I}(1 - M)_I \epsilon_{a_R}, \tag{16}$$

and this result leads to

THEOREM 3. *If $H(I)$ holds, $SS(I)/\sigma^2 = SS^*(I)$ and is distributed as χ^2 with $(t-1)_I$ d.f. and is independent of $SS(R)$ provided $H(R)$ holds.*

Proof. This theorem is an immediate consequence of (16) and Theorem 2.

It follows, if we assume that the highest order interactions, $\mu(R, a_R)$, are zero, that $F(I) = MS(I)/MSE$ has the F-distribution with $(t-1)_I$ and $(t-1)_R$ d.f., where $MS(I) = SS(I)/(t-1)_I$, $MSE = SS(R)/(t-1)_R$. This result makes it possible to test the hypothesis $H(I)$ at the $100(1-\alpha)\%$ level of significance by rejecting $H(I)$ if $F(I) > F_\alpha$, where $\Pr(F > F_\alpha) = \alpha$. This test seems reasonable if we recall ([1]) that

$$E(MSE) = \sigma^2, \qquad E[MS(I)] = \sigma^2 + t_R \sigma^2(I)/t_I,$$

where $\sigma_2(I)$ is not a variance but is defined by $\sigma^2(I) = t_I M_I M_I \mu^2(I, a_I)/(t-1)_I$.

Suppose we wish to test the hypothesis that both $H(I)$ and $H(J)$ are true while making our usual assumption that $H(R)$ holds. Then $SS(I)$ and $SS(J)$ are distributed as $\chi^2\sigma^2$, independent of each other and of $SSE = SS(R)$, with $(t-1)_I$ and $(t-1)_J$ d.f., respectively. Hence their sum is distributed as $\chi^2\sigma^2$, independent of SSE, with $(t-1)_I + (t-1)_J$ d.f. Thus $F(I, J) = MS(I, J)/MSE$ has the F-distribution with $(t-1)_I + (t-1)_J$ and $(t-1)_R$ d.f., where $SS(I, J) = SS(I) + SS(J)$ is the *pooled* sum of squares, corresponding to the two hypotheses being tested, with degrees of freedom obtained by pooling the corresponding degrees of freedom, and $MS(I, J) = SS(I, J)/[(t-1)_I + (t-1)_J]$.

There may be cases where not only the $(r-1)$st order interactions, $\mu(R, a_R)$, are assumed to be zero but also certain others, say the $\mu(J, a_J)$. The preceding argument leads to replacing $SSE = SS(R)$ by $SSE_1 = SS(J) + SS(R)$ with a corresponding pooling of the degrees of freedom.

5. The r-way crossed classification with replication. We return to the notation of Section 3 and find that, with minor modifications, the results of Section 4 still hold. We note that $\mu_a = E(X_a) = \mu_{a_R}$ and that, in Theorem 2, we should extend the range of k to $r+1$. Also (14) becomes

$$\mu(I, a_I) = M_{R+1-I}(1 - M)_I \mu_a, \tag{17}$$

a result based on the assumption that

$$\mu_a = \mu_{a_R} = (1 + D)\mu(R + 1, a_{R+1}). \tag{18}$$

But then $\mu(I, r+1; a_I, a_{r+1}) = M_{R-I}(1 - M_{r+1})(1 - M)_I \mu_a = 0$, since $(1 - M_{r+1})\mu_a = (1 - M_{r+1})\mu_{a_R} = 0$, and it follows that (4) and (18) are consistent. We conclude that any interaction, whose last index is $r+1$, is *a priori* zero. We have seen in the preceding section that, in this case, the error sum of squares, SSE, is the sum of all sums of squares with last index $r+1$. This is simply the definition of SSE given in (10). Theorem 3 leads us to the conclusion that $SS(I)$ and SSE are independently distributed as $\chi^2\sigma^2$ with $(t-1)_I$ and $t_R(t_{r+1} - 1)$ d.f. Thus we may still use the test of $H(I)$ given in Section 4.

6. Computation of the sums of squares. We first establish

THEOREM 4. $t_I \sum_{a_I} T_{a_I}^2/t = (1+D)_I SS(I)$, where $SS(0) = t\overline{X}^2$ and $T_a = \sum_b (\delta_{ab})_I X_b$.

Proof. Consider the set of t_I variables $\overline{X}_{a_I}$ and apply Theorem 1 and its second corollary to obtain

$$\sum_{a_I} \overline{X}_{a_I}^2 = (1 + D)_I SS(I) = t_I M_{b_I} M_{c_I} [(1 + D)_I (t\delta_{bc} - 1)_I] \overline{X}_{b_I} \overline{X}_{c_I}$$

$$= t_I M_b M_c [(1 + D)_I (t\delta_{bc} - 1)_I] X_b X_c = (t_I/t)(1 + D)_I S\overline{S}(I).$$

Then, since $\overline{X}_{a_I} = t_I T_{a_I}/t$, the desired result follows.

In order to obtain the entries in the analysis of variance table, we first compute the *correction term* $SS(0) = T^2/t$, where T is the sum of all the observations. Then, by Theorem 4,

$$SS(i) = (t_i/t) \sum_{a_i=1}^{t_i} T_{a_i}^2 - T^2/t \qquad (i = 1, \cdots, r).$$

Next,

$$(t_i t_j/t) \sum_{a_i=1}^{t_i} \sum_{a_j=1}^{t_j} T_{a_i a_j}^2 = SS(i, j) + SS(i) + SS(j) + SS(0)$$

and we can solve for $SS(i, j)$. Continuing this procedure, we can compute all of the sums of squares save SSE and SST. It is easy to verify that $SST = \sum_a X_a^2 - T^2/t$ and then SSE may be obtained by subtraction since Corollary 1 of Theorem 1 holds.

The reader who is interested in further applications of this operational method will find them in another paper [1]. Models other than the Type I model of this paper are considered, expected values of mean squares are obtained, and the variances of variance components are determined.

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SOME INEQUALITIES FOR POLYNOMIALS*

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1. Let $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ be an arbitrary polynomial of degree n in $z = x + iy$ and let $|p(z)| \leqq M$ on a set $\mathcal{E}$. The modulus of $p'(z)$ on $\mathcal{E}$ has an upper bound depending on M, on n, and on the set $\mathcal{E}$. In this connection Sewell [3] has proved the following theorem.

THEOREM A. *Let $p(z)$, a polynomial of degree n in z, be in modulus less than or equal to a constant M on a set $\mathcal{E}$ which has no isolated points and whose complement has finite connectivity. Then $|p'(z)| \leqq MK_1(\mathcal{E})n^2$ on $\mathcal{E}$, where the constant $K_1(\mathcal{E})$ depends only on $\mathcal{E}$.*

This theorem generalizes the following theorem of A. Markoff (see [2]).

* Research supported by the National Science Foundation.

THEOREM B. *Let* $|p(x)| \leqq M$ *in the interval* $-1 \leqq x \leqq 1$. *Then* $|p'(x)| \leqq Mn^2$ *for* $-1 \leqq x \leqq 1$.

The following theorem due to Bernstein (see [2]) gives another estimate for $|p'(x)|$.

THEOREM C. *Let* $|p(x)| \leqq M$ *in the interval* $-1 \leqq x \leqq 1$. *Then* $|p'(x)| \leqq Mn/(1-x^2)^{1/2}$ *for* $-1 < x < 1$.

For every subinterval of $(-1, 1)$ this result is very much better than the theorem of Markoff if n is sufficiently large.

It was later on proved by Sewell [4] that if the set $\mathcal{E}$ is bounded by an analytic Jordan curve $\mathcal{C}$ then $|p'(z)| \leqq MK_2(\mathcal{C})n$, where $K_2(\mathcal{C})$ is a constant depending only on $\mathcal{C}$. If $\mathcal{C}$ is the unit circle, we know by a theorem of Bernstein that $K_2(\mathcal{C}) = 1$. It may be interesting to consider some well-known sets $\mathcal{E}$ and to solve the problem in each case.

We prove first of all the following theorem

THEOREM 1. *If* $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ *is an arbitrary polynomial of degree* n *in* z *and* $\mathcal{C}$ *is the boundary of an ellipse in the* z*-plane then for every* $k \geqq 1$

$$\left(\int_{\mathcal{C}} |p'(z)|^k |dz|\right)^{1/k} \leqq \frac{n}{b}\left(\int_{\mathcal{C}} |p(z)|^k |dz|\right)^{1/k},$$

where b *denotes the semiminor axis of the ellipse.*

Proof. Let us suppose that the ellipse has its foci in the points c and $-c$ on the real axis. The Laurent polynomial

$$p\left[\frac{c}{2}\left\{\frac{b+(b^2+c^2)^{1/2}}{c}\zeta + \frac{c}{b+(b^2+c^2)^{1/2}}\frac{1}{\zeta}\right\}\right] \tag{1}$$

takes on $|\zeta| = 1$ the same set of values as $p(z)$ does on the boundary $\mathcal{C}$ of the ellipse. Hence for $k \geqq 1$

$$\begin{aligned}
&\int_{\mathcal{C}} |p'(z)|^k |dz| \\
&= \int_{|\zeta|=1} \left|\frac{d}{d\zeta} p\left[\frac{c}{2}\left\{\frac{b+(b^2+c^2)^{1/2}}{c}\zeta + \frac{c}{b+(b^2+c^2)^{1/2}}\frac{1}{\zeta}\right\}\right]\frac{d\zeta}{dz}\right|^k \left|\frac{dz}{d\zeta}\right| |d\zeta| \\
&= \int_{|\zeta|=1} \left|\frac{d}{d\zeta} p\left[\frac{c}{2}\left\{\frac{b+(b^2+c^2)^{1/2}}{c}\zeta + \frac{c}{b+(b^2+c^2)^{1/2}}\frac{1}{\zeta}\right\}\right]\right|^k \left|\frac{d\zeta}{dz}\right|^{k-1} |d\zeta| \\
&= \int_{|\zeta|=1} \left|\frac{d}{d\zeta} p\left[\frac{c}{2}\left\{\frac{b+(b^2+c^2)^{1/2}}{c}\zeta + \frac{c}{b+(b^2+c^2)^{1/2}}\frac{1}{\zeta}\right\}\right]\right|^k \\
&\qquad \cdot \left|\frac{c}{2}\left\{\frac{b+(b^2+c^2)^{1/2}}{c} - \frac{c}{b+(b^2+c^2)^{1/2}}\frac{1}{\zeta^2}\right\}\right|^{1-k} |d\zeta|
\end{aligned}$$

$$\leqq \int_{|\zeta|=1} \left| \frac{d}{d\zeta} p\left[\frac{c}{2} \left\{ \frac{b+(b^2+c^2)^{1/2}}{c} \zeta + \frac{c}{b+(b^2+c^2)^{1/2}} \frac{1}{\zeta} \right\} \right] \right|^k \cdot \left[\frac{2\{b+(b^2+c^2)^{1/2}\}}{\{b+(b^2+c^2)^{1/2}\}^2 - c^2} \right]^{k-1} |\, d\zeta \,|$$

$$= \frac{1}{b^{k-1}} \int_{|\zeta|=1} \left| \frac{d}{d\zeta} p\left[\frac{c}{2} \left\{ \frac{b+(b^2+c^2)^{1/2}}{c} \zeta + \frac{c}{b+(b^2+c^2)^{1/2}} \frac{1}{\zeta} \right\} \right] \right|^k |\, d\zeta \,|.$$

A theorem of Zygmund [6] states that if $s_n(z) = \sum_{-n}^{n} b_\nu z^\nu$, then for every $k \geqq 1$

$$\left(\int_{|z|=1} |\, s_n'(z) \,|^k |\, dz \,| \right)^{1/k} \leqq n \left(\int_{|z|=1} |\, s_n(z) \,|^k |\, dz \,| \right)^{1/k}.$$

Applying this result to the Laurent polynomial (1), we get from above for $k \geqq 1$

$$\int_{\mathcal{C}} |\, p'(z) \,|^k |\, dz \,|$$

$$\leqq \frac{n^k}{b^{k-1}} \int_{|\zeta|=1} \left| p\left[\frac{c}{2} \left\{ \frac{b+(b^2+c^2)^{1/2}}{c} \zeta + \frac{c}{b+(b^2+c^2)^{1/2}} \frac{1}{\zeta} \right\} \right] \right|^k |\, d\zeta \,|$$

$$= \frac{n^k}{b^{k-1}} \int_{\mathcal{C}} |\, p(z) \,|^k \left| \frac{d\zeta}{dz} \right| |\, dz \,| \leqq \frac{n^k}{b^k} \int_{\mathcal{C}} |\, p(z) \,|^k |\, dz \,|,$$

and the theorem follows by taking the kth root of both the sides.

The case $k = \infty$ is $\max_{z \in \mathcal{C}} |\, p'(z) \,| \leqq (n/b) \max_{z \in \mathcal{C}} |\, p(z) \,|$, also proved by Sewell [5].

By applying Zygmund's theorem to $p(c_1\zeta + c_0 + c_{-1}\zeta^{-1})$, where c_1, c_0 and c_{-1} are complex numbers, analogous results for various other curves may be deduced. We also prove

THEOREM 2. *If $\mathcal{E}$ is the arc of the parabola $x^2 = ay (a > 0)$ with endpoints $(-1, 1/a)$, $(1, 1/a)$, and if $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n in z and $|\, p(z) \,| \leqq M$ on $\mathcal{E}$, then*

$$|\, p'(z) \,| \leqq \max \left[\frac{4Mn^2 a}{\{a^2 + 4[1 - (4n^2)^{-1}]\}^{1/2}},\ 2Mn \right]. \tag{2}$$

Proof. The polynomial $p(x + i(x^2/a))$ is of degree $2n$ and takes in the interval $-1 \leqq x \leqq 1$ the same set of values as $p(z)$ does on the arc of the parabola. Hence from Theorem B it follows that for $z \in \mathcal{E}$, $|\, 1 + 2i(x/a) \,|\, |\, p'(z) \,| \leqq 4Mn^2$ or

$$|\, p'(z) \,| \leqq \frac{4Mn^2 a}{(a^2 + 4x^2)^{1/2}}. \tag{3}$$

If we apply Theorem C to the polynomial $p(x+i(x^2/a))$ we get

$$|p'(z)| \leqq \frac{2Mna}{(a^2+4x^2)^{1/2}(1-x^2)^{1/2}}. \tag{4}$$

Case i. If $a \geqq 2$, the term on the right side of (4) increases with $|x|$ and therefore it follows from (4) that for $x^2 \leqq 1-(4n^2)^{-1}$

$$|p'(z)| \leqq \frac{4Mn^2a}{\{a^2+4[1-(4n^2)^{-1}]\}^{1/2}}. \tag{5}$$

On the other hand the term on the right side of (3) decreases as $|x|$ increases and so from (3) we have (5) for $x^2>1-(4n^2)^{-1}$.

Case ii. $a<2$. As in Case i, (5) holds for $|x|>[1-(4n^2)^{-1}]^{1/2}$. As $|x|$ increases from 0 to $[\frac{1}{8}(4-a^2)]^{1/2}$, the term on the right side of (4) decreases and attains a minimum at $|x|=[\frac{1}{8}(4-a^2)]^{1/2}$. For $|x|>[\frac{1}{8}(4-a^2)]^{1/2}$, it is monotone increasing. Therefore, for $|x| \leqq [1-(4n^2)^{-1}]^{1/2}$

$$|p'(z)| \leqq \max\left[p'\{[1-(4n^2)^{-1}]^{1/2}\}, p'(0)\right],$$

which is equal to the right side of (2) and the theorem is proved.

2. Let $p(z)=\sum_{\nu=0}^{n} a_\nu z^\nu$ be a polynomial of degree n such that $|p(z)| \leqq 1$ for $|z| \leqq 1$. If $R>1$ then by the maximum principle

$$\max_{|z|=1/R} |z^n p(z^{-1})| \leqq \max_{|z|=1} |z^n p(z^{-1})| = \max_{|z|=1} |p(z^{-1})| = \max_{|z|=1} |p(z)| \leqq 1$$

so that $\max_{|z|=1/R} |p(z^{-1})| \leqq R^n$. But $\max_{|z|=1/R} |p(z^{-1})| \equiv \max_{|z|=R} |p(z)|$ and therefore $\max_{|z|=R} |p(z)| \leqq R^n$.

The example $p(z)=z^n$ shows that the estimate is best possible. Thus, the upper bound of $|p(z)|$ over a certain circle being given, we can determine its upper bound over a larger concentric circle.

An upper bound for $|p(z)|$ over the linear interval $(-1, 1)$ having been given, Bernstein determined the upper bound for $|p(z)|$ over an ellipse having foci at the points 1 and -1. He proved ([1], vol. 1, p. 137, problem III 271):

THEOREM D. *Let $p(z)$ be a polynomial of degree n or less such that in the interval $(-1, 1)$ of the real axis $|p(z)| \leqq 1$. If R is the sum of the semiaxes of an ellipse passing through the point z and having foci at the points -1 and 1, then $|p(z)| \leqq R^n$.*

Here we prove

THEOREM 3. *Let $p(z)$ be a polynomial of degree n. If $\mathcal{C}$ denotes the boundary of an ellipse whose foci lie at the points -1 and 1 and the sum of whose semiaxes is R, then for a positive integer m,*

$$\int_{\mathcal{C}} |p(z)|^m |dz| \leqq 2R^{mn+1} \int_{-1}^{1} |p(x)|^m dx. \tag{6}$$

The proof makes use of the following lemma.

LEMMA. *Let* $f(z)=\sum_{\nu=-M}^{N} a_\nu z^\nu$, *where* M *and* N *are positive integers. If* $\mu(r, f)$ *denotes the mean value of* $|f(z)|$ *over the circle* $|z|=r$, *then*

$$\mu(r_1, f)/r_1^N \geqq \mu(r_2, f)/r_2^N, \qquad 0 < r_1 < r_2.$$

Proof. $F(z)=z^{-N}f(z)$ is regular in the punctured plane $|z|>0$, the point $z=\infty$ included. $\mathfrak{F}(z)=F(z^{-1})$ is regular in the finite circle $|z|<\infty$ and $\mu(r, F(z))$ is the mean value of $|\mathfrak{F}(z)|$ on the circumference $|z|=r^{-1}$. Consequently

$$\mu(r_1^{-1}, \mathfrak{F}(z)) \geqq \mu(r_2^{-1}, \mathfrak{F}(z)), \qquad r_2^{-1} < r_1^{-1} < \infty,$$

or $\mu(r_1, F(z)) \geqq \mu(r_2, F(z))$, $0<r_1<r_2$, from which the lemma follows.

Proof. If m is a positive intger, then

$$\left\{p\left(\frac{\zeta+\zeta^{-1}}{2}\right)\right\}^m (1-\zeta^{-2})$$

is a function of the form $f(\zeta)=\sum_{\nu=-M}^{N} a_\nu \zeta^\nu$ with $M=mn+2$, $N=mn$. Therefore, it follows from the lemma that

$$\begin{aligned}\int_C |p(z)|^m |dz| &= \int_{|\zeta|=R} \left|p\left(\frac{\zeta+\zeta^{-1}}{2}\right)\right|^m \frac{1}{2} |1-\zeta^{-2}| \, |d\zeta| \\ &\leqq R^{mn+1} \int_0^{2\pi} |p(\cos\theta)|^m |\sin\theta| \, d\theta \\ &= 2R^{mn+1} \int_0^{\pi} |p(\cos\theta)|^m \sin\theta d\theta = 2R^{mn+1} \int_{-1}^{1} |p(x)|^m dx.\end{aligned}$$

By taking the mth root of both the sides of (6) and making $m\to\infty$ we get Theorem D.

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MEAN VALUE THEOREMS FOR FUNCTIONS WITH FINITE DERIVATES

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The law of the mean and many of its extensions require the existence of the derivative. However, since the derivative may fail to exist, it seems desirable to have expressions which may serve us when there is no derivative. The purpose of this paper is to consider extensions of the law of the mean to functions which have finite right and left "Dini derivates." Theorems of Karamata [1] and Vučković [4] as corrected by Utz [3] are contained in our results.

In some cases we need not require that all derivates be finite. Since these cases are easily resolved, they will be left for the consideration of the reader.

Let f be a function defined on a finite closed interval $a \leqq x \leqq b$. Throughout this paper we will use D^+f, D_+f, D^-f, and D_-f to denote the upper right, lower right, upper left, and lower left derivates ([2], p. 188), respectively, of f. In addition, if p and q are real numbers, we will use $pD(+)f+qD(-)f$ to denote any one of the expressions

$$p_1D^+f + q_1D^-f, \qquad p_2D^+f + q_2D_-f, \qquad p_3D_+f + q_3D^-f, \qquad p_4D_+f + q_4D_-f.$$

Finally, we will use $pD(\overset{+}{-})f+qD(\overset{-}{+})f$ to denote all four of the above expressions.

Our first theorem is an extension of Rolle's theorem.

THEOREM 1. *Let f be a function which is continuous at each point of the finite closed interval $a \leqq x \leqq b$, let $f(a)=f(b)=0$ and let D^+f, D_+f, D^-f, and D_-f be finite at each point of the open interval $a<x<b$. Then there exist numbers ξ, p, and q with $a<\xi<b$, $p \geqq 0$, $q \geqq 0$, $p+q=1$ for which $pD(+)f(\xi)+qD(-)f(\xi)=0$.*

Proof. To avoid unnecessary repetition we shall prove the theorem for only one possible choice of derivates. The relations for other pairs of derivates can be similarly proved.

If $f(x) \equiv 0$ on $a \leqq x \leqq b$, then we may take $\xi=\frac{1}{2}(a+b)$ and $p=q=\frac{1}{2}$. If $f(x) \not\equiv 0$ on $a \leqq x \leqq b$, then, since f is continuous, it assumes a positive maximum or a negative minimum or both on $a<x<b$. Let ξ, $a<\xi<b$, be such that f assumes a maximum or a minimum at ξ. If $D^+f(\xi)=0$, let $p=1$ and $q=0$. If $D^+f(\xi) \neq 0$, then the product $D^+f(\xi)D^-f(\xi) \leqq 0$ and we may choose

$$p = \frac{D^-f(\xi)}{D^-f(\xi) - D^+f(\xi)} \geqq 0, \qquad q = \frac{D^+f(\xi)}{D^+f(\xi) - D^-f(\xi)} \geqq 0,$$

which completes the proof.

Theorem 2 is an extension of the law of the mean. Its proof, as well as the proofs of our remaining theorems, is based on Theorem 1.

* The author is indebted to the referee for his helpful suggestions.

THEOREM 2. *Let f be a function which is continuous at each point of the finite closed interval $a \leqq x \leqq b$ and let the derivates D^+f, D_+f, D^-f, and D_-f be finite at each point of the open interval $a < x < b$. Then there exist numbers ξ, p, and q with $a < \xi < b$, $p \geqq 0$, $q \geqq 0$, $p+q=1$ for which*

$$f(b) - f(a) = [b - a][pD(+)f(\xi) + qD(-)f(\xi)].$$

Proof. We shall prove the theorem for only one possible choice of derivates. The other cases can be proved in a similar manner. Let

$$F(x) = [b - a][f(x) - f(a)] - [x - a][f(b) - f(a)].$$

The function F is continuous on $a \leqq x \leqq b$, F has finite one-sided derivates on $a < x < b$, and $F(a) = F(b) = 0$. Hence F satisfies the conditions of Theorem 1 and there exist numbers ξ, p, and q with $a < \xi < b$, $p \geqq 0$, $q \geqq 0$, $p+q=1$ such that $pD^+F(\xi) + qD^-F(\xi) = 0$. But

$$D^+F(x) = [b - a]D^+f(x) - [f(b) - f(a)],$$
$$D^-F(x) = [b - a]D^-f(x) - [f(b) - f(a)],$$

and the required result now follows.

Our next theorem is an extension of Taylor's formula with remainder.

THEOREM 3. *Let the function f, defined on the closed interval $a \leqq x \leqq b$, be such that its first $n-1$ derivatives exist and are continuous on $a \leqq x \leqq b$ (only right and left derivatives are required at a and b, respectively). In addition, let f be such that the derivates $D^+f^{(n-1)}$, $D_+f^{(n-1)}$, $D^-f^{(n-1)}$, and $D_-f^{(n-1)}$ of its $(n-1)$st derivative $f^{(n-1)}$ are finite at each point of the open interval $a < x < b$. Then for any number v such that $n - v > 0$ there exist numbers ξ, p, and q with $a < \xi < b$, $p \geqq 0$, $q \geqq 0$, $p+q=1$ for which*

$$f(b) = f(a) + (b - a)f'(a) + \cdots + \frac{(b - a)^{n-1}f^{(n-1)}(a)}{(n - 1)!}$$
$$+ \frac{(b - a)^{n-v}(b - \xi)^v}{(n - v)(n - 1)!}[pD(+)f^{(n-1)}(\xi) + qD(-)f^{(n-1)}(\xi)].$$

Proof. We shall prove the theorem for only one possible choice of derivates. The relations for other pairs of derivates can be similarly proved. Let

$$F(x) = -f(b) + f(x) + (b - x)f'(x) + \cdots$$
$$+ \frac{(b - x)^{n-1}}{(n - 1)!}f^{(n-1)}(x) + \frac{(b - x)^{n-v}}{(b - a)^{n-v}}K,$$

where

$$K = f(b) - f(a) - (b - a)f'(a) - \cdots - \frac{(b - a)^{n-1}}{(n - 1)!}f^{(n-1)}(a).$$

Since F satisfies the conditions of Theorem 1, there exist numbers ξ, p, and q with $a<\xi<b$, $p\geqq 0$, $q\geqq 0$, $p+q=1$ such that $pD^+F(\xi)+qD^-F(\xi)=0$. But

$$D^+F(x) = \frac{(b-x)^{n-1}}{(n-1)!}D^+f^{(n-1)}(x) - \frac{(n-v)(b-x)^{n-v-1}}{(b-a)^{n-v}}K,$$

$$D^-F(x) = \frac{(b-x)^{n-1}}{(n-1)!}D^-f^{(n-1)}(x) - \frac{(n-v)(b-x)^{n-v-1}}{(b-a)^{n-v}}K,$$

and the required result then follows.

Let f and g be functions which are continuous at each point of the closed interval $a\leqq x\leqq b$ and which have finite derivates D^+f, D_+f, D^-f, D_-f, and D^+g, D_+g, D^-g, D_-g, respectively, at each point of the open interval $a<x<b$. We shall say that $D(+)f+D(-)f$ is partially bounded above by $D(+)g+D(-)g$ provided there exist numbers ξ, p, and q with $a<\xi<b$, $p\geqq 0$, $q\geqq 0$, $p+q=1$ such that

$$[g(b)-g(a)][pD(+)f(\xi)+qD(-)f(\xi)] \leqq [f(b)-f(a)][pD(+)g(\xi)+qD(-)g(\xi)].$$

Moreover, we shall say that the four expressions $D(\overset{+}{+})f+D(\overset{-}{-})f$ are partially bounded above by $D(+)g+D(-)g$ provided each of the expressions is partially bounded above by $D(+)g+D(-)g$. The notion of partially bounded below is defined by replacing *above* by *below* and $\leqq$ by $\geqq$ in the preceding definition.

Our final results, which are extensions of Cauchy's generalized law of the mean, will be stated as three related parts of Theorem 4. We shall prove only one relation from Part (a). The other relations can be similarly proved.

THEOREM 4. *Suppose that the functions f and g are continuous at each point of the closed interval $a\leqq x\leqq b$ and that the one-sided upper and lower derivates of f and g are finite at each point of the open interval $a<x<b$.*

(a) *If $f(b)-f(a)\geqq 0$, $g(b)-g(a)\geqq 0$, then $D(\overset{+}{+})f+D(\overset{-}{-})f$ is partially bounded above by D^+g+D^-g and partially bounded below by D_+g+D_-g. Moreover, D^+f+D_-f and D_+f+D^-f are partially bounded above and below by D^+g+D_-g and D_+g+D^-g, respectively.*

(b) *If $f(b)-f(a)\leqq 0$, $g(b)-g(a)\leqq 0$, then $D(\overset{+}{+})f+D(\overset{-}{-})f$ is partially bounded above by D_+g+D_-g and partially bounded below by D^+g+D^-g. Moreover, D^+f+D_-f and D_+f+D^-f are partially bounded above and below by D^+g+D_-g and D_+g+D^-g, respectively.*

(c) *If $f(b)-f(a)\leqq 0$, $g(b)-g(a)\geqq 0$, then $D(\overset{+}{+})f+D(\overset{-}{-})f$ and $D(\overset{+}{+})g+D(\overset{-}{-})g$ are partially bounded above by D_+g+D_-g and D^+f+D^-f, respectively, and partially bounded below by D^+g+D^-g and D_+f+D_-f, respectively. Moreover, D^+f+D_-f and D_+f+D^-f are partially bounded above and below by D_+g+D^-g and D^+g+D_-g, respectively.*

Proof (a). Let $F(x)=[g(b)-g(a)][f(x)-f(a)]-[f(b)-f(a)][g(x)-g(a)]$. Since F satisfies the conditions of Theorem 1, there exist numbers ξ, p, q, with

$a<\xi<b$, $p\geqq 0$, $q\geqq 0$, $p+q=1$, such that $pD_+F(\xi)+qD_-F(\xi)=0$. But ([2], p. 190)

$$D_+F(x) \leqq [g(b)-g(a)]D^+f(x) - [f(b)-f(a)]D^+g(x),$$
$$D_-F(x) \leqq [g(b)-g(a)]D^-f(x) - [f(b)-f(a)]D^-g(x).$$

It follows that

$$[g(b)-g(a)][pD^+f(\xi)+qD^-f(\xi)] \geqq [f(b)-f(a)][pD^+g(\xi)+qD^-g(\xi)],$$

which was to be shown.

Added in proof. The reader will find of interest a paper by S. R. Mukhoti, *On a generalization of some theorems of the differential calculus*, Bull. Calcutta Math. Soc., vol. 50, 1958, pp. 87–93.

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DIRECT PROOF OF LANCZOS' DECOMPOSITION THEOREM

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In his recent paper [1], C. Lanczos has proved the following general

THEOREM. *For every real $m\times n$ matrix A of rank $p>0$ it is possible to find an $n\times p$ column-orthogonal matrix Y and an $m\times p$ column-orthogonal matrix X such that*

$$A = X\Lambda Y', \tag{1}$$

*where Λ is a $p\times p$ diagonal matrix of rank p.**

While the inherent interest of Lanczos' construction of the column-orthogonal matrices X, Y cannot be denied, it seems to be desirable to have a more direct proof of the theorem. Some new applications are discussed in the later sections of the present paper.

1. We interpret the matrix A as representative of a linear mapping of the real n-dimensional vector space R_n into the space R_m, more accurately: onto a

* The prime indicates transposition. For all notations see the author's book [2].—Theorem and proof remain valid in the case of complex matrices if one replaces "orthogonal" by "unitary-(or hermitian-) orthogonal", X' by $\bar{X}'=X^*$, etc.

p-dimensional linear subspace L_p of R_m. We show first that in R_n there is an orthonormal system of vectors $y^{(1)}, \cdots, y^{(p)}$ such that their images in R_m, *i.e.* the vectors

$$Ay^{(1)} = z^{(1)}, \cdots, Ay^{(p)} = z^{(p)}$$

form an orthogonal basis of L_p. Indeed, let us choose $y^{(1)}, \cdots, y^{(p)}$ as the first p column vectors of the orthogonal $n \times n$ matrix

$$\mathfrak{Y} = (Y, Y_0), \quad Y = y^{(1)}, \cdots, y^{(p)}, \quad Y_0 = (y^{(p+1)}, \cdots, y^{(n)}),$$

which reduces the $n \times n$ Gram matrix $A'A$ (positive definite if $p = n$, semidefinite if $p < n$) by orthogonal congruence transformation into its normal form

$$\mathfrak{Y}'A'A\mathfrak{Y} = (A\mathfrak{Y})'A\mathfrak{Y} = \begin{pmatrix} \mu_1 & & & & & \\ & \ddots & & & 0 & \\ & & \mu_p & & & \\ & & & 0 & & \\ & 0 & & & \ddots & \\ & & & & & 0 \end{pmatrix} \qquad (\mu_i > 0)$$

where $\mu_1, \cdots, \mu_p$ are the nonzero eigenvalues of $A'A$.† Hence the vectors $z^{(1)}, \cdots, z^{(p)}$ form an orthogonal system:

$$z^{(i)\prime}z^{(j)} = \begin{cases} 0 & \text{if } j \neq i \\ \mu_i & \text{if } j = i \end{cases} \qquad (i, j = 1, \cdots, p).$$

Further let $\lambda_i = \sqrt{\mu_i}$ and $z^{(i)} = \lambda_i x^{(i)}$. Then $x^{(1)}, \cdots, x^{(p)}$ form an orthonormal vector system in R_m and

$$AY = (\lambda_1 x^{(1)}, \cdots, \lambda_p x^{(p)}) = X\Lambda, \tag{2}$$

where

$$X = (x^{(1)}, \cdots, x^{(p)}), \qquad \Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_p \end{pmatrix}.$$

Since in general Y is an $n \times p$ nonsquare column-orthogonal matrix, (2) is not immediately equivalent to (1). But we observe that $y^{(p+1)}, \cdots, y^{(n)}$ are the eigenvectors of $A'A$ for the eigenvalue 0, *i.e.*, a complete system of linearly independent solutions of the linear system $A'Ay = 0$. As there are also $n - p$ (and not more) linearly independent solutions of the system $Ay = 0$, and these obviously satisfy the former system, it follows that

$$Ay^{(p+1)} = 0, \cdots, Ay^{(n)} = 0 \quad \text{or} \quad AY_0 = 0. \tag{3}$$

† The matrices A and $A'A$ always have the same rank p; *cf.* [2], p. 142, Theorem 19.5.

Therefore $A\mathfrak{Y}=A(Y,\ Y_0)=(AY,\ 0)=(X\Lambda,\ 0)$ and

$$A = (X\Lambda, 0)\mathfrak{Y}' = (X\Lambda, 0)\begin{pmatrix} Y' \\ Y_0' \end{pmatrix} = X\Lambda Y',$$

i.e., the relation (1).

Since $X'X$ is equal to the $p\times p$ unit matrix, one has immediately from (2)

$$X'AY = \Lambda. \tag{4}$$

This may be considered as the generalized "principal-axes transformation" of an arbitrary real matrix A. As pointed out by Lanczos it neglects the eigenvalue zero.

Applying transposition to (1) and to (4) we obtain the relations

$$A' = Y\Lambda X', \tag{1}'$$

$$Y'A'X = \Lambda, \tag{4}'$$

showing that X and Y are interchanged if A is replaced by A'. Thus X is the $m\times p$ matrix whose columns represent a system of orthonormal eigenvectors of AA' which belong to the nonzero eigenvalues of AA'.

Also X may be completed to an $m\times m$ orthogonal matrix $\mathfrak{X}=(X,\ X_0)$, where $A'X_0=0$ (*cf.* [3]) in the same way as Y to an $n\times n$ orthogonal matrix. We note that because $\mathfrak{X}$ and $\mathfrak{Y}$ are orthogonal matrices

$$X'X_0 = 0, \qquad X_0'X = 0; \qquad Y'Y_0 = 0, \qquad Y_0'Y = 0. \tag{5}$$

2. Assuming that for a given matrix A the matrices X, Y, have been determined one can obtain an explicit solution of the linear system $Ay=b$. This has been dealt with by Lanczos [1]. It is briefly resumed here in order to make the connection with a treatment given elsewhere (*cf.* [3], [4]).

With regard to (1) the given system may be written in the form

$$X\Lambda Y'y = b. \tag{6}$$

If a solution y exists then by (5) $X_0'b=0$ and this condition is also known to be sufficient. Thus b is orthogonal to the column vectors of X_0 and is therefore a linear combination of the column vectors of X, *i.e.*, $b=Xb^{(0)}$ so that $b^{(0)}=X'b$ and the system (6) is reduced to $Y'y=\Lambda^{-1}b^{(0)}$.

Using unknown column vectors

$$u = \begin{pmatrix} u_1 \\ \cdot \\ \cdot \\ \cdot \\ u_p \end{pmatrix}, \qquad u^{(0)} = \begin{pmatrix} u_{p+1} \\ \cdot \\ \cdot \\ \cdot \\ u_n \end{pmatrix},$$

we may put

$$v = Yu + Y_0u^{(0)} = u_1y^{(1)} + \cdots + u_py^{(p)} + u_{p+1}y^{(p+1)} + \cdots + u_ny^{(n)}. \tag{7}$$

With regard to (5) it follows that $Y'y = u + Y'Y_0u^{(0)} = u = \Lambda^{-1}b^{(0)}$. Thus $u^{(0)}$ remains unrestricted. By (7) one has then the general solution of (6) in the form

$$y = Y\Lambda^{-1}X'b + Y_0u^{(0)}. \tag{8}$$

The vector $Y\Lambda^{-1}X'b$ represents the "orthogonal solution" (*cf.* [3], [4]) which is orthogonal to every solution $Y_0u^{(0)} = v$ of the homogeneous system $Ay = 0$. This is at the same time the solution vector y of minimum length: $y'y = \min$.

3. Now let us assume that the linear system $Ay = b$ has no solution, *i.e.*, $X_0'b \neq 0$. Instead of asking for a solution one may then ask for a vector y such that the deviation vector $Ay - b$ has minimum length, *i.e.*,

$$\delta^2 = (Ay - b)'(Ay - b) = \min.$$

For A we use again the "orthogonal representation" (1) and for abbreviation we put $Y'A'b = d$, $Y'y = z$. Making use of a device of [5] we can write the sum of the squares of the deviation components in the form

$$\begin{aligned}\delta^2 &= z'\Lambda^2z - z'd - d'z + b'b \\ &= z'\Lambda^2z - z'\Lambda^2\Lambda^{-2}d - d'\Lambda^{-2}\Lambda^2z + b'b \\ &= (z - \Lambda^{-2}d)'\Lambda^2(z - \Lambda^{-2}d) - d'\Lambda^{-2}d + b'b.\end{aligned}$$

Thus δ^2 will assume its minimum value

$$\delta^2_{\min} = b'b - d'\Lambda^{-2}d = b'b - b'X\Lambda Y'Y\Lambda^{-2}Y'Y\Lambda X'b = b'(E - XX')b$$

if and only if $z = \Lambda^{-2}d = \Lambda^{-2}Y'A'b$, which by (1)' is equal to $\Lambda^{-1}X'b$.

It remains to solve the system $Y'y = z$. Making use of (7) as in Section 2 we obtain

$$y = Yz + Y_0u^{(0)} = Y\Lambda^{-1}X'b + Y_0u^{(0)},$$

where $u^{(0)}$ is arbitrary. This is an explicit solution of the central problem in the "method of least squares," valid in all cases. It has the same form as the solution (8) in the case of solvability of the linear system $Ay = b$.

We observe that in the case of nonsolubility the vector b cannot be written in the form $Xb^{(0)}$; thus $b = Xb^{(0)} + X_0b^{(1)}$, where $b^{(1)} \neq 0$ because $X_0'b \neq 0$. Hence one obtains the minimum deviation vector

$$\begin{aligned}Ay - b &= X\Lambda Y'(Y\Lambda^{-1}X'Xb^{(0)} + Y\Lambda^{-1}X'X_0b^{(1)}) - Xb^{(0)} - X_0b^{(1)} \\ &= Xb^{(0)} + 0 - Xb^{(0)} - X_0b^{(1)} = -X_0b^{(1)},\end{aligned}$$

and, with regard to the column-orthogonality of X_0: $\delta^2_{\min} = b^{(1)\prime}b^{(1)}$.

4. We conclude with a few examples.

(a) Extending Lanczos' example (*cf.* [1], p. 672) let A be a matrix of rank $p = 1$. Then there are two nonzero column vectors a, b such that $A = ab'$ and its orthogonal representation according to the theorem will be

$$A = \frac{a}{\|a\|} \Lambda \frac{b'}{\|b\|}, \qquad \Lambda = \|a\|\|b\|$$

if $\|a\| = \sqrt{(a'a)}$. Indeed the Gram matrix $A'A = a'a \cdot bb'$ has only one eigenvalue different from zero, its trace $a'a \cdot b'b = \|a\|^2\|b\|^2$.

The linear system $ab'y = c$ has a solution y if and only if $c = \gamma a$, where γ is a number. By (8) one has

$$y = \frac{b}{\|b\|} \frac{1}{\|a\|\|b\|} \frac{a'}{\|a\|} \gamma a + v = \gamma \frac{b}{\|b\|^2} + v,$$

where v is any vector perpendicular to b.

In the case of nonsolvability any vector y yielding minimum deviation is represented by

$$y = \frac{a'c}{\|a\|^2\|b\|^2} b + v$$

and the minimum deviation vector,

$$Ay - c = \frac{a'c}{a'a} a - c,$$

does not depend on b. The square of its length equals

$$\delta^2_{\min} = \frac{a'a \cdot c'c - (a'c)^2}{a'a}.$$

(b) Let A be a skew-symmetric matrix of rank 2. With regard to [2], p. 153, there are two linearly independent vectors a, b in R_n such that

$$A = ba' - ab'.$$

The Gram matrix

$$A'A = b'b \cdot aa' + a'a \cdot bb' - a'b(ba' + ab')$$

has a and b as eigenvectors, both with the eigenvalue $\mu = a'a \cdot b'b - (a'b)^2 = \lambda^2 > 0$ (by Cauchy-Schwarz's inequality). Thus

$$y^{(1)} = \frac{a}{\|a\|}, \qquad y^{(2)} = \frac{a'a \cdot b - a'b \cdot a}{\|a\|\lambda}$$

form a system of orthonormal eigenvectors of $A'A$. Now, since $Ay^{(1)} = \lambda y^{(2)}$, $Ay^{(2)} = -\lambda y^{(1)}$, one has $x^{(1)} = y^{(2)}$, $x^{(2)} = -y^{(1)}$ and it is readily verified that if $X = (y^{(2)}, -y^{(1)})$, $Y = (y^{(1)}, y^{(2)})$ then

$$X'AY = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \Lambda = \lambda E$$

and also

$$(9) \qquad A = X\Lambda Y' = (\lambda y^{(2)}, -\lambda y^{(1)}) \begin{pmatrix} y^{(1)\prime} \\ y^{(2)\prime} \end{pmatrix} = \lambda(y^{(2)}y^{(1)\prime} - y^{(1)}y^{(2)\prime}).$$

We consider again the linear system $Ay=c$. It has a solution y if and only if $c=Xc^{(0)}=c_1^0x^{(1)}+c_2^0x^{(2)}=c_1^0y^{(2)}-c_2^0y^{(1)}$. By (8)

$$\begin{aligned} y &= Y\begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda^{-1} \end{pmatrix} X'Xc^{(0)} + v \\ &= \lambda^{-1}(c_1^0 y^{(1)} + c_2^0 y^{(2)}) + v \\ &= \lambda^{-1}\left(c_1^0 \frac{a}{\|a\|} + c_2^0 \frac{a'a\cdot b - a'b\cdot a}{\|a\|\lambda}\right) + v. \end{aligned}$$

In the case of nonsolvability, that is if the three vectors a, b, c are linearly independent, the y yielding minimum deviation is given by

$$\begin{aligned} y &= Y\begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda^{-1} \end{pmatrix} X'c + v \\ &= \lambda^{-1}(y^{(1)}y^{(2)\prime} - y^{(2)}y^{(1)\prime})c + v = -\lambda^{-2}Ac + v. \end{aligned}$$

The minimum deviation itself is $Ay-c=-\lambda^{-2}(A^2+\lambda^2E)c$ and

$$\delta_{\min}^2 = \lambda^{-4}c'(A^2+\lambda^2E)^2c.$$

Now by (9)

$$A^2 = -\lambda^2(y^{(1)}y^{(1)\prime} + y^{(2)}y^{(2)\prime}), \qquad A^4 = \lambda^4(y^{(1)}y^{(1)\prime}+y^{(2)}y^{(2)\prime}) = -\lambda^2A^2.$$

Therefore $(A^2+\lambda^2E)^2=\lambda^2A^2+\lambda^4E=\lambda^2(A^2+\lambda^2E)$ and

$$\begin{aligned} \delta_{\min}^2 &= \lambda^{-2}c'(A^2+\lambda^2E)c \\ &= \lambda^{-2}(\lambda^2c'c + 2b'c\cdot c'a\cdot a'b - a'a(b'c)^2 - b'b(a'c)^2) \\ &= \lambda^{-2}\begin{vmatrix} a'a & a'b & a'c \\ b'a & b'b & b'c \\ c'a & c'b & c'c \end{vmatrix}. \end{aligned}$$

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THE DIAGONALIZATION OF CERTAIN NORMAL MATRICES

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1. Introduction. A numerical method for the determination of the characteristic values of arbitrary matrices was recently developed by the author, and is described in references [1] and [2]. The basis of this method is a theorem of I. Schur, which states that any arbitrary matrix A of complex elements may be transformed by means of unitary transformations into a triangular matrix having the characteristic values of A along its diagonal. The method proceeds as follows: An arbitrary off-diagonal nonzero element $a_{ij} \equiv b$ is chosen to be the "pivot" of a unitary transform U which differs from the unit matrix only in the positions (i, i), (i, j), (j, i), and (j, j). These positions in U are taken, respectively, by the elements $t = r \exp(i\theta)$, $-(1-r^2)^{1/2}$, $(1-r^2)^{1/2}$, and $\bar{t}$. The quantities r, $0 < r < 1$, and θ are to be determined in such a manner as to minimize the difference

$$M_1 - M = \sum_{\substack{i,j=1 \\ i<j}}^{n} (|a_{ij}^{(1)}|^2 - |a_{ij}|^2),$$

where $A_1 \equiv (a_{ij}^{(1)}) = U^{-1} A U$. It turns out that

$$M_1 - M < H(R, \theta) = \sum_{s=1}^{4} C_{s-1}(\theta) R^s, \tag{1}$$

where $R = r^{-1}(1-r^2)^{1/2}$ and the $C_s(\theta)$ are well-defined expressions in θ involving certain elements of A. The determination of R and θ on the basis of $M_1 - M \leqq \min H(R, \theta)$ leads to the norm-reducing cubic polynomial

$$F(R, \theta) \equiv 4C_3R^3 + 3C_2R^2 + 2C_1R + C_0 = 0. \tag{2}$$

For matrices that are symmetric this method reduces to the well-known and widely used procedure due to Jacobi. Now further investigation of this method has led to additional interesting results, some of which are described below. It was found, namely, that certain classes of normal matrices can be reduced by this method to diagonal form by means of a finite number of transformations only.

2. Arbitrary normal matrices of order 2. Let us consider

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \tag{3}$$

where the elements are arbitrary complex numbers. The assumption that A is normal, *i.e.*, $AA^{CT} = A^{CT}A$ (with A^{CT} denoting the conjugate transpose of A), leads to the two conditions

$$|b| = |c|, \tag{4}$$

$$b(\bar{a} - \bar{d}) = \bar{c}(a - d). \tag{5}$$

Putting $a=|a|\exp(i\alpha)$, $b=|b|\exp(i\beta)$, $c=|c|\exp(i\gamma)$, $d=|d|\exp(i\delta)$, $a-d=|a-d|\exp(i\epsilon)$, it is seen that (5) leads to

$$2\epsilon = \beta + \gamma. \tag{6}$$

Let us now construct the norm-reducing polynomial (2). From [1],

$$C_3 = |c|^2,$$

$$C_2 = -2|c|\,[|d|\cos(\theta+\gamma-\delta) - |a|\cos(\theta+\gamma-\alpha)],$$

$$C_1 = |d-a|^2 - 2|bc|\cos(2\theta-\beta+\gamma) + \sum_{i+1\le k\le j-1}[|a_{ki}|^2 + |a_{jk}|^2],$$

$$C_0 = 2|b|\,[|d|\cos(\theta+\delta-\beta) - |a|\cos(\theta+\alpha-\beta)] + 2\sum_{i+1\le k\le j-1}[|a_{ik}|\,|a_{jk}|\cos(\theta+\alpha_{jk}-\alpha_{ik}) - |a_{ki}|\,|a_{kj}|\cos(\theta+\alpha_{ki}+\alpha_{kj})].$$

Choosing $\theta=\frac{1}{2}(\beta-\gamma)$ and utilizing (4), (5), and (6), we obtain for the matrix A

$$C_3 = |b|^2, \qquad C_2 = -2|b|\,[|d|\cos(\epsilon-\delta) - |a|\cos(\epsilon-\alpha)],$$

$$C_1 = |d-a|^2 - 2|b|^2, \qquad C_0 = 2|b|\,[|d|\cos(\epsilon-\delta) - |a|\cos(\epsilon-\alpha)].$$

However, expansion shows that

$$|d|\cos(\epsilon-\delta) - |a|\cos(\epsilon-\alpha) = |d-a|. \tag{7}$$

Consequently,

$$F(R,\theta) = 4|b|^2R^3 - 6|b|\,|d-a|R^2 + 2(|d-a|^2 - 2|b|^2)R + 2|b|\,|d-a| = 2[|b|R^2 - |d-a|R - |b|][2|b|R - |d-a|].$$

We choose then for R one of the roots of

$$|b|R^2 - |d-a|R - |b| = 0 \tag{8}$$

and for the transformation variable z, the number $R\exp(i\theta)$, where $\theta=\frac{1}{2}(\beta-\gamma)$. In the transformed matrix

$$A_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix},$$

by [1], $b_1=-\bar{t}^2[cz^2-(d-a)z-b]$, $c_1=-t^2[b\bar{z}^2-(d-a)\bar{z}-c]$, where $t=r\exp(i\theta)$, $r^2=(1+R^2)^{-1}$. But

$$cz^2 - (d-a)z - b = \exp(i\beta)[|b|R^2 - |d-a|R - |b|],$$

$$b\bar{z}^2 - (d-a)\bar{z} - c = \exp(i\gamma)[|b|R^2 - |d-a|R - |b|],$$

so that $b_1=c_1=0$. Thus A_1 is diagonalized, with a_1 and d_1 representing the characteristic values of A.

3. Another class of normal matrices. Let us next consider the matrix

$$B = \begin{bmatrix} u & v & w \\ w & u & v \\ v & w & u \end{bmatrix}, \tag{9}$$

where $u = u_1 + iu_2$, v, w are arbitrary complex numbers. Clearly, B is normal: $B^{CT}B = BB^{CT}$. The matrices B are circulants; such matrices are known to have characteristic values, all of which are expressible in closed form in terms of the primitive roots of unity. What is considered of interest here is the fact that the method of [2] diagonalizes these matrices in exactly three well-defined unitary transformations. Causey [3] stated that it takes four sweeps, *i.e.* twelve transformations to triangularize a matrix of the type (9) by Greenstadt's method.

Let us now turn to the diagonalization of B by the method of [2].

Step 1. We choose first $\theta_1 = 0$, $i = 1$, $j = 3$, and put

$$U_1 = \begin{bmatrix} r_1 & 0 & -s_1 \\ 0 & 1 & 0 \\ s_1 & 0 & r_1 \end{bmatrix}. \tag{10}$$

Consequently,

$$B_1 \equiv U_1^{-1}BU_1 = \begin{bmatrix} u + r_1s_1(v + w) & r_1v + s_1w & r_1^2w - s_1^2v \\ r_1w + s_1v & u & r_1v - s_1w \\ r_1^2v - s_1^2w & r_1w - s_1v & u - r_1s_1(v + w) \end{bmatrix}. \tag{11}$$

The value of r_1 will be appropriately determined later on.

Step 2. Let us choose next in (11), $\theta_2 = 0$, $i = 1$, $j = 2$, or

$$U_2 = \begin{bmatrix} r_2 & -s_2 & 0 \\ s_2 & r_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{12}$$

The value of r_2 is to be fixed in such a manner as to reduce the norm $\sum_{i<j} |a_{ij}^{(2)}|^2$ of the super-diagonal elements $a_{ij}^{(2)}$ in $B_2 = U_2^{-1}B_1U_2$, as outlined in [2]. A short calculation shows that, for this step, the coefficients of the norm-reducing cubic are

$$\begin{aligned} C_3 &= r_1^2|w|^2 + s_1^2|v|^2 + 2r_1s_1(v_1w_1 + v_2w_2), \\ C_2 &= 2r_1s_1[r_1|w|^2 + s_1|v|^2 + (r_1 + s_1)(v_1w_1 + v_2w_2)], \\ C_1 &= r_1^2s_1^2|v + w|^2 - 2[r_1s_1(|v|^2 + |w|^2) + (v_1w_1 + v_2w_2)], \\ C_0 &= -2r_1s_1[s_1|w|^2 + r_1|v|^2 + (r_1 + s_1)(v_1w_1 + v_2w_2)]. \end{aligned}$$

These expressions become greatly simplified for the choice

$$r_1 = s_1 \tag{13}$$

or $r_1 = 2^{-\frac{1}{4}}$. In that case we have

$$C_3 = \tfrac{1}{2}\,|\,v + w\,|^2, \qquad C_2 = (\tfrac{1}{2})^{1/2}\,|\,v + w\,|^2,$$
$$C_1 = -\tfrac{3}{4}\,|\,v + w\,|^2, \qquad C_0 = -(\tfrac{1}{2})^{1/2}\,|\,v + w\,|^2.$$

The cubic polynomial (2) may now be factored into

$$|\,v + w\,|^2[2^{1/2}R^2 + r - 2^{1/2}][2^{1/2}R + 2^{-1}], \tag{14}$$

leading to the roots $R_2 = 2^{-\frac{1}{4}}, -2^{\frac{1}{4}}, -(2\cdot 2^{\frac{1}{4}})^{-1}$. Of these roots R_2, that particular value is to be chosen which makes

$$H(R, \theta) \equiv C_3R^4 + C_2R^3 + C_1R^2 + C_0R \tag{15}$$

a minimum. Thus we take $R_2 = r_1 = 2^{-1/2}$. Then $r_2 = (\frac{2}{3})^{1/2}$ and, consequently,

$$B_2 \equiv U_2^{-1}B_1U_2 = \begin{bmatrix} u+v+w & 0 & 0 \\ 0 & u - {}^{1/2}(v+w) & \tfrac{1}{2}3^{1/2}(v-w) \\ 0 & -\tfrac{1}{2}3^{1/2}(v-w) & u - \tfrac{1}{2}(v+w) \end{bmatrix}. \tag{16}$$

The super-diagonal norm has decreased from $\frac{1}{2}|v+w|^2+\frac{3}{4}|v-w|^2$ in B_1 to $\frac{3}{4}|v-w|^2$ in B_2, *i.e.*, by the norm $|b|^2$ of the pivot b in B_1.

Step 3. We now choose $i=2, j=3$, and put $\theta = \frac{1}{2}(\beta-\gamma)$. Since $\beta = \gamma+\pi$, $\theta = \frac{1}{2}\pi$, (2) becomes $R^2 = 1$, so that $R_3 = 1$, whence $r_3 = 2^{-1/2}$, $s_3 = i2^{-1/2}$, and

$$U_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & i2^{-1/2} & -2^{-1/2} \\ 0 & 2^{-1/2} & -i2^{-1/2} \end{bmatrix}. \tag{17}$$

It is seen that

$$B_3 \equiv U_3^{-1}B_2U_3 = \begin{bmatrix} u+v+w & 0 & 0 \\ 0 & u-\tfrac{1}{2}v+w+i3^{1/2}(v-w) & 0 \\ 0 & 0 & u-\tfrac{1}{2}v+w-i3^{1/2}(v-w) \end{bmatrix}. \tag{18}$$

The diagonalization of B has thus been accomplished by means of the unitary transformation $B_3 = U^{-1}BU$, where

$$U = U_1U_2U_3 = \begin{bmatrix} 3^{-1/2} & -\tfrac{1}{2} - i\tfrac{1}{2}3^{-1/2} & \tfrac{1}{2}3^{-1/2} + i\tfrac{1}{2} \\ 3^{-1/2} & i3^{-1/2} & -3^{-1/2} \\ 3^{-1/2} & \tfrac{1}{2} - i\tfrac{1}{2}3^{-1/2} & \tfrac{1}{2}3^{-1/2} - i\tfrac{1}{2} \end{bmatrix}. \tag{19}$$

The diagonal elements of B_3 are, naturally, the characteristic values of B.

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ON THE METHOD OF CONTRACTANTS

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The history of the recently reported "method of contractants" ([2], [3]) is much older than is apparently appreciated. A paper by Charles L. Dodgson (Lewis Carroll) [1] appears to contain all the essential information of the papers by Macmillan [2] and Lotkin [3]. In fact, it goes rather further into the matter of transposing to remove internal zeros in such a manner as to retain, as useful, most of the quantities previously calculated.

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1. Charles L. Dodgson, On condensation of determinants, Proc. Roy. Soc. London, vol. 15, 1866, pp. 150–155.
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MATHEMATICAL NOTES

EDITED BY ROY DUBISCH, Fresno State College

Because of the large number of papers on hand, consideration of new papers for this department has been temporarily suspended.

HOW TO PLAY BASEBALL

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1. Introduction. This is supposed to be a mathematics paper, our preposterous title notwithstanding. Now there is very little which mathematics can do about certain aspects of the great American game; we do not expect to make any vital contributions to the axioms "Swing hard," "Pitch good," "Run fast," "Sleep lots," "Don't drink," "Play heads up," etc.

But there is one facet of baseball which does lend itself to the intrusion of

the mathematical egghead, and this is the guessing game between the pitcher and batter, the game of "Should I put it in there?" and "Should I swing at the next one?."

To give a "game-theoretic" solution to this little guessing game, we must first strip to its bare essentials, and in fact we shall even idealize these bare essentials.

Thus, although in the real game the location of men on base, the score, the number of the inning, and the number of outs all should be counted in as "state variables," we will not do so. Furthermore, we also ignore the possibility of extra base hits—all this of course in order to obtain a model which is computationally feasible. To be sure, on larger computing machines more and more realistic models can be analysed, but for now we are content with the following.

2. Idealized rules.

1. *Simultaneously*, the pitcher decides whether to pitch a ball (B) or a strike (S), and the batter decides whether to swing (s) or take (t).

2. If the joint decision is, then the batter gets a

B	t	Ball
B	s	Strike
S	t	Strike
S	s	Single

3. The count is as usual

3 strikes before 4 balls	you're out
4 balls before 3 strikes	you walk

If we concentrate, for the moment, on the pitcher's strategy, we see that, depending on the count, he must make a random choice for his next pitch, random but not necessarily 50-50. Thus his decision, when the count is $m: n$ (m balls, n strikes $0 \leqq m < 4$, $0 \leqq n < 3$) is given by a probability $p = p_{m,n}$, $0 \leqq p \leqq 1$, where p is his probability of throwing a strike. The complete strategy for the pitcher, then, consists in the knowledge of the 12 numbers $p_{m,n}$.

A similar remark is valid for the batter whose strategy will be completely known when 12 numbers $q_{m,n}$ (probability of swinging when count is $m: n$) are found.

Our problem, then, is the computation of these numbers. Let us note that a straightforward matrix approach, which is usually tried in game theory, leads to a matrix of size approximately 128 by 128. We must seriously think of another approach, and it turns out that some ideas in dynamic programming* are what click here.

3. The new approach.

Pitcher's strategy. We concentrate first on the *value* of the game rather than

* R. Bellman, Dynamic Programming, Princeton, 1957.

on the $p_{m,n}$. To this end we define $V_{m,n}$ to be the probability, given that the count is $m:n$ and that we are employing an *optimal strategy*, of striking the batter out.

Note that $V_{4,n}=0$, $n<3$ and $V_{m,3}=1$, $m<4$. These determine our "boundary values" for $V_{m,n}$. We now develop a recurrence relation.

Suppose the count is $m:n$ and that our optimal strategy calls for a strike to be pitched $p_{m,n}$ of the time. Then, if the batter swings, our new value becomes

$$A = 0\cdot p_{m,n} + V_{m,n+1}(1 - p_{m,n})$$

whereas, if he takes, our new value becomes

$$B = V_{m,n+1}p_{m,n} + V_{m+1,n}(1 - p_{m,n}).$$

Note that the minimum of these two expressions is what we should expect as our new value! In game theory we must always assume that the opponent knows our strategy and combats it as much as possible. Furthermore, it was assumed that $p_{m,n}$ was optimal; therefore, $p_{m,n}$ maximizes min $[A, B]$. Furthermore, min $[A, B]$ is maximized by that value of p which forces $A = B$, that is, by

$$p_{m,n} = \frac{V_{m,n+1} - V_{m+1,n}}{2V_{m,n+1} - V_{m+1,n}} \tag{1}$$

and so, substituting back,

$$\min [A, B] = \frac{V_{m,n+1}^2}{2V_{m,n+1} - V_{m+1,n}}.$$

But $\min[A, B] = V_{m,n}$ and

$$V_{m,n} = \frac{V_{m,n+1}^2}{2V_{m,n+1} - V_{m+1,n}}.$$

This is the required recurrence relation.

Let us now review these results. We have

$$V_{m,n} = \frac{V_{m,n+1}^2}{2V_{m,n+1} - V_{m+1,n}}, \qquad V_{4,n} = 0 \ (n < 3), \qquad V_{m,3} = 1 \ (m < 4),$$

which allows us to compute all the $V_{m,n}$. We have also seen, as a by-product, that (1) holds. Consequently, having all the $V_{m,n}$ we can compute all the $p_{m,n}$. This, the reader will recall, gives us the entire strategy!

4. Example. Find $p_{2,2}$. From

$$p_{22} = \frac{V_{23} - V_{32}}{2V_{23} - V_{32}} = \frac{1 - V_{32}}{2 - V_{32}},$$

we see we need only find V_{32}. But,

$$V_{32} = \frac{V_{33}^2}{2V_{33} - V_{42}} = \frac{1}{2\cdot 1 - 0} = \frac{1}{2},$$

and so $p_{22} = (1-\frac{1}{2})/(2-\frac{1}{2}) = \frac{1}{3}$. Thus, when the count is 2:2, pitch a strike with probability $\frac{1}{3}$.

Batter's strategy. By setting up the same conditions for the batter, we find that the equations become identical, and we conclude that $q_{m,n} = p_{m,n}$. Therefore, a knowledge of the $p_{m,n}$ is all that is required.

5. Computations. We now give the approximate values obtained for some of the $p_{m,n}$.

	0	1	2	3	m
0	.300	.347	.410	.5	
1	.262	.312	.385	.5	
2	.2	.25	.333	.5	
n					

The exact values are rather complicated fractions, *e.g.*, $p_{20} = \frac{89}{217}$.

One further computation is of interest. We find that $V_{0,0} = .4995$, and so a one-bagger is just about as likely as an out. From this we can find the expected number of runs per inning. The result is .75 runs.

ENUMERATION OF THE POSITIVE RATIONALS

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Let $p_1, p_2, \cdots$ be the primes in increasing order. We define a 1-1 mapping of the positive integers onto the positive rationals by

$$G\colon\ n = \prod_1^\infty p_i^{a_i} \xrightarrow{G} r = \prod_1^\infty p_i^{b_i},$$

where $b_i = a_{2i-1}$ if $a_{2i-1} \neq 0$ and $b_i = -a_{2k}$ if $a_{2i-1} = 0$ and is the kth such a_{2j-1}.

But this seems a bit cumbersome. A more elegant and general set of mappings may be described as follows: Let f be a 1-1 mapping of the positive integers and 0 onto the positive and negative integers such that $f(0) = 0$. Then

$$F\colon\ n = \prod_1^\infty p_i^{a_i} \xrightarrow{F} r = \prod_1^\infty p_i^{f(a_i)}$$

is a 1-1 mapping of the positive integers onto the positive rationals. The obvious choice for f is $f(n) = (-1)^{n+1}[\frac{1}{2}(n+1)]$ where $[x]$ is the greatest integer function.

Thus the enumeration of the positive rationals is reduced to the enumeration of the integers.

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A PROPERTY OF THE SALTUS OPERATOR

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Let f be a bounded real valued function defined on the interval $[0, 1]$. The saltus operator S is defined by the formula

$$(1) \qquad Sf(x) = \limsup_{\epsilon \to 0} \{ |f(y) - f(z)| : |x - y| < \epsilon, |x - z| < \epsilon \}.$$

Thus the saltus operator maps the set of all real bounded functions into itself. In this note we shall prove a conjecture of S. K. Stein, namely, that S satisfies the relationship $S^2 = S^3$; *i.e., for any real valued bounded function f defined on* $[0, 1]$ *we have* $S[Sf] = S[S[Sf]]$.

While this relationship is easily proved, and is possibly well known, it does not seem to be stated explicitly in the literature. The result will be demonstrated with the help of three well-known facts which will be stated as lemmas.

LEMMA 1. *If f is an arbitrary bounded function, defined on* $[0, 1]$, *then Sf is an upper semicontinuous function* ([1], p. 95).

LEMMA 2. *The points of discontinuity of a semicontinuous real function defined on an interval form a set of the first category* ([1], p. 110). *Hence every subinterval of* $[0, 1]$ *must contain a point of continuity.*

LEMMA 3. *Let f be a bounded real function defined on* $[0, 1]$. *Then f is continuous at $x \in [0, 1]$ if and only if* $Sf(x) = 0$ ([1], p. 96).

Using the fact that $S^2 f = g$ is a nonnegative function which is equal to zero on a set which is dense in $[0, 1]$, we have for any $x \in [0, 1]$

$$\begin{aligned} Sg(x) &= \limsup_{\epsilon \to 0} \{ |g(y) - g(z)| : |x - y| < \epsilon, |x - z| < \epsilon \} \\ &= \limsup_{\epsilon \to 0} \{ g(y) | : |x - y| < \epsilon \}. \end{aligned}$$

Using the upper semicontinuity of g we have

$$\limsup_{\epsilon \to 0} \{ g(y) | : |x - y| < \epsilon \} = g(x).$$

We have now shown $Sg = g$, which is the desired result.

Reference

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ON THE METHOD OF VARIATION OF PARAMETERS—II

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In [1], the method of variation of parameters was modified and applied to linear differential equations of second order. Here, this method is modified and applied to linear difference equations of second order:

$$y(n+2)+Q(n)y(n+1)+P(n)y(n)=R(n), \tag{1}$$

where $P(n)\neq 0$ and $R(n)\not\equiv 0$ for $n\geqq 0$. Let the complementary solution, $y_c(n)$ be given by $y_c(n)=au(n)+bv(n)$, where $u(n)$ and $v(n)$ are two linearly independent solutions of (1) with $R(n)\equiv 0$. Let the particular integral, $y_p(n)$, be given by $y_p(n)=A(n)u(n)+B(n)v(n)$, where $A(n)$ and $B(n)$ are functions of n to be determined. Then the method of variation of parameters yields the system of two equations [2],

$$u(n+1)\Delta A(n)+v(n+1)\Delta B(n)=0, \tag{2}$$

$$u(n+2)\Delta A(n)+v(n+2)\Delta B(n)=R(n). \tag{3}$$

Since the Wronskian, $W(n)=u(n)v(n+1)-u(n+1)v(n)\neq 0$ for $n\geqq 0$, we find that

$$y_p(n)=-u(n)\sum_{k=0}^{n-1}\frac{v(k+1)R(k)}{W(k+1)}+v(n)\sum_{k=0}^{n-1}\frac{u(k+1)R(k)}{W(k+1)}. \tag{4}$$

It is recalled that (2) is arbitrary, that (3) is determined from (1), and that the general solution of (1) is $y(n)=y_c(n)+y_p(n)$. We note that (2) and (3) remain the same if $y_p(n)=[A(n)+\alpha]u(n)+[B(n)+\beta]v(n)$, where α and β are arbitrary constants. In practice, one chooses $\alpha=\beta=0$.

Let $f(n)\equiv f^*(n, u(n), v(n))$ be defined for $n\geqq 0$. If we introduce $f(n)$ into the righthand side of (2), then (3) assumes a new form. Assuming a particular integral, $Y_p(n)=[A(n)+\alpha]u(n)+[B(n)+\beta]v(n)$, where α and β are constants whose values will be specified later, the new system of equations is

$$u(n+1)\Delta A(n)+v(n+1)\Delta B(n)=f(n), \tag{5}$$

$$u(n+2)\Delta A(n)+v(n+2)\Delta B(n)=R(n)-f(n+1)-Q(n)f(n). \tag{6}$$

We will now show that (5) and (6), accompanied with a proper specification of α and β, yields the particular integral $Y_p(n)\equiv y_p(n)$, as given by (4).

Since $u(n)$ and $v(n)$ satisfy (1) with $R(n)\equiv 0$, we find that

$$W(n+1)\Delta A(n)=-v(n+1)R(n)+v(n+1)f(n+1)-P(n)v(n)f(n),$$

$$W(n+1)\Delta B(n)=u(n+1)R(n)-u(n+1)f(n+1)+P(n)u(n)f(n),$$

$$Y_p(n)=y_p(n)+g(n),$$

where

$$g(n)=u(n)\sum_{k=0}^{n-1}\frac{v(k+1)f(k+1)-P(k)v(k)f(k)}{W(k+1)} -v(n)\sum_{k=0}^{n-1}\frac{u(k+1)f(k+1)-P(k)u(k)f(k)}{W(k+1)}+\alpha u(n)+\beta v(n). \tag{7}$$

Given $f(n)$, there exist suitable choices of α and β such that $g(n)\equiv 0$. If $f(n)\equiv 0$,

then $g(n)\equiv 0$ if $\alpha=\beta=0$. We show now that there exists no $f(k)\not\equiv 0$ such that $v(k+1)f(k+1)-P(k)v(k)f(k)\equiv 0$ and $u(k+1)f(k+1)-P(k)u(k)f(k)\equiv 0$ simultaneously. Suppose now that such a $f(k)\not\equiv 0$ exists. Since $W(k+1)\equiv P(k)W(k)$, we have $v(k)f(k)\equiv C_1W(k)$ and $u(k)f(k)\equiv C_2W(k)$, where $C_1\neq 0$, $C_2\neq 0$ are fixed constants. From this we have that $C_1u(k)\equiv C_2v(k)$, which contradicts the assumption that $u(k)$ and $v(k)$ are linearly independent solutions of (1).

Recalling the formula for summation by parts,

$$\sum_{k=0}^{n-1}\sigma(k)\Delta H(k)=[\sigma(k)H(k)]_0^n-\sum_{k=0}^{n-1}H(k+1)\Delta\sigma(k),$$

we have

$$\sum_{k=0}^{n-1}\frac{\Delta[v(k)f(k)]}{W(k+1)}=\left[\frac{v(k)f(k)}{W(k+1)}\right]_0^n-\sum_{k=0}^{n-1}\frac{[1-P(k+1)]v(k+1)f(k+1)}{W(k+2)}. \tag{8}$$

Since $v(k+1)f(k+1)-P(k)v(k)f(k)=\Delta[v(k)f(k)]+(1-P(k))v(k)f(k)$, we have, using (8),

$$\begin{aligned}\sum_{k=0}^{n-1}&\frac{\Delta[v(k)f(k)]+(1-P(k))v(k)f(k)}{W(k+1)}\\ &=\left[\frac{v(k)f(k)}{W(k+1)}\right]_0^n-\sum_{k=0}^{n-1}\Delta\left[\frac{(1-P(k))v(k)f(k)}{W(k+1)}\right]\\ &=\left[\frac{v(k)f(k)-(1-P(k))v(k)f(k)}{W(k+1)}\right]_0^n=\left[\frac{v(k)f(k)}{W(k)}\right]_0^n.\end{aligned} \tag{9}$$

Since (9) holds if $v(k)$ is replaced by $u(k)$, we have

$$g(n)=\alpha u(n)+\beta v(n)+u(n)\cdot\left[\frac{v(k)f(k)}{W(k)}\right]_0^n-v(n)\cdot\left[\frac{u(k)f(k)}{W(k)}\right]_0^n. \tag{10}$$

Thus, $g(n)=\{\alpha-[v(0)f(0)/W(0)]\}u(n)+\{\beta+[u(0)f(0)/W(0)]\}v(n)$. Hence, if $\alpha=v(0)f(0)/W(0)$, $\beta=-u(0)f(0)/W(0)$, then $g(n)\equiv 0$. If $f(0)=0$, then $\alpha=\beta=0$. This can always be done by replacing $f(n)$ by $[f(n)-f(0)]$ in (5) and (6). We note that $u(0)$ and $v(0)$ cannot both be zero for then $W(0)=0$, $W(k)\equiv 0$. In only two cases, when $f(n)\equiv 0$ or $f(n)\not\equiv 0$, $f(0)=0$, can one choose $\alpha=\beta=0$.

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A NEW PLANET

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1. In our times the conception of a new planet is no longer a fantasy *à la* Jules Verne. Such a body is, of course, subjected to the fundamental theorems of mechanics and moves according to Kepler's laws. The planets in existence may be divided into two classes, the *inner* and the *outer* ones, and whatever the differences in their appearance from a geocentric point of view may be, they have in common that their orbits on the celestial sphere are rather complicated; all of them have a period of *direct* motion and a stage in which they are *retrograde*. We consider now a body which is a *new* planet in the sense that it does not belong to either of the two classes because its distance to the sun is the same as the earth's. We simplify the matter by assuming that the orbits in space of all planets are circles (but we allow them to be in different planes) and by neglecting their mutual influences. We ask for the apparent motion of the new planet when it is observed from a terrestial standpoint.

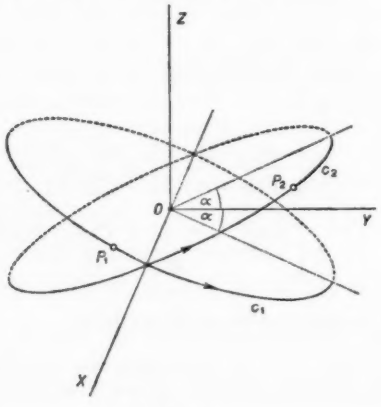

FIG. 1

2. We consider two planets, P_1 (the earth) and P_2 (the new planet) which move in concentric circular orbits about O (the sun) with the same radius R, lying in different planes V_1 and V_2, the angle between which is 2α $(0 \leqq \alpha < \frac{1}{2}\pi)$ (Fig. 1). According to Kepler's laws both planets move with uniform velocity and, moreover, this is the same for both; the latter circumstance is the reason why the relative motion of P_1 and P_2 is rather simple. The two circles c_1 and c_2 have two points of intersection S_1 and S_2. We assume that the two circular motions have a phase-difference so that collision is excluded. Let this difference be 2ψ $(0 < \psi < \pi)$; that means that if P_1 is in S_1, the arc S_1P_2 is 2ψ.

We introduce a cartesian frame as follows. OS_1 is taken as the x-axis, and the bisecting planes of V_1 and V_2 are $y=0$ and $z=0$, such that, seen from the

point (0, 0, 1), the planets are both circulating in the left-hand direction. If ω is the common angular velocity and $t=0$ the moment when both arcs P_1S_1 and S_1P_2 are equal to ψ, the coordinates of P_1 and P_2 are

$$(1)\quad \begin{aligned} x_1 &= R\cos(\omega t-\psi), \quad y_1 = R\sin(\omega t-\psi)\cos\alpha, \quad z_1 = -R\sin(\omega t-\psi)\sin\alpha,\\ x_2 &= R\cos(\omega t+\psi), \quad y_2 = R\sin(\omega t+\psi)\cos\alpha, \quad z_2 = R\sin(\omega t+\psi)\sin\alpha. \end{aligned}$$

The components of the vector P_1P_2 are

$$(2)\quad \begin{aligned} p_1 &= x_2 - x_1 = -2R\sin\omega t\sin\psi,\\ p_2 &= y_2 - y_1 = 2R\cos\alpha\cos\omega t\sin\psi,\\ p_3 &= z_2 - z_1 = 2R\sin\alpha\sin\omega t\cos\psi. \end{aligned}$$

From this it follows immediately that $\sin\alpha\cos\psi\cdot p_1+\sin\psi\cdot p_3=0$, which means that the direction of P_1P_2 is always parallel to the plane through the y-axis with the equation

$$(3)\qquad \sin\alpha\cos\psi\cdot x+\sin\psi\cdot z=0.$$

Hence, seen from the earth, *the new planet moves along a great circle on the celestial sphere.*

The plane V_1 has the equation $\sin\alpha\cdot y+\cos\alpha\cdot z=0$. Therefore, if δ is the acute angle between V and V_1, we have

$$(4)\qquad \cos^2\delta = (\cos^2\alpha\sin^2\psi)/(\sin^2\alpha\cos^2\psi+\sin^2\psi),$$

$$(5)\qquad \operatorname{tg}\delta = (\operatorname{tg}\alpha)/(\sin\psi).$$

Hence *the angle δ between the new planet's apparent orbit and the ecliptic is given by* (5) and it is easily seen that the angle between the x-axis and the line of intersection of V and V_1 is $\frac{1}{2}\pi-\psi$.

3. The fact that all vectors P_1P_2 are parallel to a fixed plane gives rise to the question: What is the locus of the line P_1P_2 in space? The points P_1 and P_2 move along circles which are intersecting but lie in different planes; the correspondence between the points of c_1 and c_2 is a metrically specialized projectivity. Now it is well known from line geometry that the line joining corresponding points of two projective conics generates a *quartic ruled surface.* In our case this F_4 is of a special type because of the way in which the two conics intersect. It is the type numbered VIII in the classification of Salmon.* F_4 has a double conic and a double line, which have a point in common; all generators meet both and the double line is not itself a generator. Applying this to our problem we see that in V_1 there are two generators P_1P_2, namely, when P_2 is either in S_1 or in S_2; obviously these two are parallel. Furthermore there are two generators in V_2, namely, when P_1 is in S_1 or S_2; they too are parallel. When two generators meet their point of intersection is on the double line of F_4. The conclusion is:

* G. Salmon, A Treatise on the Analytic Geometry of Three Dimensions, Dublin, 1882, p. 520.

the double line is in the plane at infinity and therefore all generators are parallel to a fixed plane. The generators in V_1 have the direction $x:y:z=\sin\psi:\cos\psi\cos\alpha:-\cos\psi\sin\alpha$, those in V_2: $x:y:z=-\sin\psi:\cos\psi\cos\alpha:\cos\psi\sin\alpha$ and the plane parallel to both is indeed $\sin\alpha\cos\psi\cdot x+\sin\psi\cdot z=0$. The equation of F_4 can be found from (1). If we put $L=\sin\alpha\cos\psi\cdot x+\sin\psi\cdot z$ and $y=y_1\cos\alpha$ it reads

$$(x^2+y_1^2)L^2-2\sin\alpha\cos\psi\cdot xLw^2+\sin^2\alpha\cdot w^2(\cos^2\psi\cdot w^2-y_1^2)=0.$$

The double line is given by $L=w=0$ and the double conic is the hyperbola $y=xL-\sin\alpha\cos\psi\cdot w^2=0$; their common point is $L=y=w=0$.

4. We return now to the problem proper and ask how the new planet moves along its great circle c lying in V. In order to study the motion in the apparent orbit we can take an arbitrary position as the initial one and we choose for this the y-axis which is in V. We determine the position of the planet by the angle ϕ which P_1P_2 makes with this axis. Then from (2)

$$\cos\phi=\frac{\cos\alpha\sin\psi\cos\omega t}{\sqrt{\{\sin^2\omega t(\sin^2\alpha\cos^2\psi+\sin^2\psi)+\cos^2\omega t\cos^2\alpha\sin^2\psi\}}} \tag{6}$$

or in view of (4)

$$\operatorname{tg}\phi=(\operatorname{tg}\omega t)/(\cos\delta). \tag{7}$$

which gives *the position of the planet as a simple function of the time.* It has the same period as the apparent motion of the sun (which was obvious beforehand); it is seen from (7) that ϕ goes from 0 to $\frac{1}{2}\pi$ if ωt does. The motion however is not uniform; for the angular velocity we have

$$\dot\phi=\frac{\omega\cos\delta}{\cos^2\delta+\sin^2\delta\sin^2\omega t}, \tag{8}$$

$\dot\phi$ always has the same sign; the new planet always goes in the same direction. *The angular velocity varies between the minimum* $\omega\cos\delta$ *and the maximum* $\omega/\cos\delta$. It is equal to ω when $\operatorname{tg}^2\omega t=\cos\delta$, that is when $\operatorname{tg}\phi=\cos^{-1/2}\delta$. When the planet passes the ecliptic we have $\omega t=-\psi$ or $\omega t=\pi-\psi$ and therefore $\operatorname{tg}\phi=-(\operatorname{tg}\psi)/(\cos\delta)$ and $\dot\phi=(\omega\cos^2\alpha)/(\cos\delta)$.

5. We consider some special cases. If $\alpha=0$ then V_1, V_2 and V coincide. The apparent motion of the new planet is along the ecliptic and with uniform angular velocity ω. The distance between P_2 and the sun, as seen from the earth, is $\frac{1}{2}\pi-\psi$ which is an arc between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$. The planet moves along the sun's orbit at a constant distance to it which corresponds to a period of at most three months on either side. Such a planet is either a morning star ($\psi<\frac{1}{2}\pi$) or an evening star ($\psi>\frac{1}{2}\pi$); for $\psi=\frac{1}{2}\pi$ it is permanently in conjunction with the sun. Another special case is given by $\psi=\frac{1}{2}\pi$, $\alpha\neq0$. Then if P_1 is in S_1, P_2 passes S_2 and *vice versa*. We have now $\delta=\alpha$ and the plane V coincides with $z=0$. On

the moments that the planet passes the ecliptic it is in conjunction with the sun and its velocity has its minimum value $\omega \cos \delta$. From (1) and (2) it follows that the planet always coincides with the sun's orthogonal projection on the plane V. In this case the double hyperbola of the quartic ruled surface degenerates into two lines. Finally we remark that a new planet's orbit may coincide with the celestial equator. Then we must have $\delta = \epsilon (= 23.5°)$ and the line of intersection of V and V_1 must pass through the equinoxes.

THE TRACES OF CERTAIN MATRICES AND CHROMATIC GRAPHS

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Let M be a symmetric matrix of order n whose diagonal elements are zero, and whose elements are either zero or unity. Let N be the matrix associated with M such that $M+N$ is a matrix whose diagonal elements are zero, but whose off-diagonal elements are unity. Denote $\operatorname{tr}(M^3)+\operatorname{tr}(N^3)$ by t.

Then we have for t the sharp lower bound

$$t \geqq \begin{cases} 2u(u-1)(u-2), & \text{if } n = 2u, \\ 4u(u-1)(4u+1), & \text{if } n = 4u+1, \\ 4u(u+1)(4u-1), & \text{if } n = 4u+3. \end{cases} \tag{1}$$

As we shall see, this result is an easy consequence of the theorem proved in [1].

Since M is an interpretation of a gathering of n people, with unity in position (ij) denoting that the ith and jth persons are acquainted and zero that they are strangers, then the elements in the diagonal of M^3 determine the number of "full triangles" (*i.e.*, mutual acquaintances involving three persons) there are in the gathering. Such triangles will be counted twice (once in each direction) from each vertex. Consequently each "full triangle" will be counted six times, and $\operatorname{tr}(M^3) = 6F$, where F is the number of full triangles. Similarly, the trace of N^3 will determine the number E of "empty triangles" (*i.e.*, mutual strangers involving three persons), and $\operatorname{tr}(N^3) = 6E$. The result (1) then quickly follows from [1].

The question raised in [1] concerning the number of full and empty polygons may possibly be answered by a close investigation of traces of other powers of M and N.

If n points are joined pairwise by lines, some of which are colored red and some colored blue, the resulting configuration is called a chromatic graph [2]. Then with an appropriate interpretation (see [1]), $\frac{1}{6}t$ is the number of solidly colored triangles in the graph. It follows that $t \leqq n(n-1)(n-2)$ and this upper bound is sharp.

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SOME PROPERTIES OF THE FIBONACCI NUMBERS

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The Fibonacci series is defined by $u_0=0$, $u_1=1$, and $u_n=u_{n-1}+u_{n-2}$. Its first few terms are: 0, 1, 1, 2, 3, 5, 8, 13 $\cdots$. The aim of this paper is to prove the

THEOREM. *If* $r>1$ *and* $n>1$, $u_{nr}>u_n^r$.

First we prove

LEMMA 1. $u_n=u_{k+1}u_{n-k}+u_ku_{n-k-1}$ $(0<k+1\leqq n)$.

We shall prove this by induction. The lemma is obvious for $k=0$ and $k=1$. If it is true for $k=i$, we have

$$u_n = u_{i+1}u_{n-i} + u_iu_{n-i-1} = u_{i+1}(u_{n-i-1} + u_{n-i-2}) + u_iu_{n-i-1}$$
$$= u_{n-i-1}(u_{i+1} + u_i) + u_{n-i-2}u_{i+1} = u_{i+2}u_{n-i-1} + u_{i+1}u_{n-i-2},$$

which completes the induction proof.

LEMMA 2. $u_{m+n}>u_mu_n$ $(m, n>0)$.

For if we substitute in the formula of Lemma 1, $m+n$ for n and m for k we have $u_{m+n}=u_{m+1}u_n+u_mu_{n-1}>u_{m+1}u_n\geqq u_mu_n$. We now have $u_{rn}>u_{(r-1)n}u_n>u_{(r-2)n}u_n^2>\cdots>u_n^r$ by repeated application of Lemma 2.

SELF-ADJOINT DIFFERENTIAL EXPRESSIONS

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The differential expression $L(y)=\sum_{i=0}^{n}A_i(x)y^{(i)}(x)$ is self-adjoint if its adjoint $M(y)=\sum_{i=0}^{n}(-1)^i(A_i(x)y(x))^{(i)}$ is identical with $L(y)$. Some properties of self-adjoint expressions are easy to develop, *e.g.*,

(a) There is no odd ordered self-adjoint expression; *i.e.*, if n is odd, then $A_n=0$.

(b) The sum (or difference) of two self-adjoint expressions is self-adjoint.

(c) If $L(y)=Ay^{(2n)}+\cdots$, and $\mathfrak{L}(y)=Ay^{(2n)}+\cdots$, are both self-adjoint, the order of $L(y)-\mathfrak{L}(y)$ is at most $2n-2$.

(d) For a given $A(x)$ (having $2n$ derivatives) there is at most one self-adjoint expression of the form

$$L(y) = Ay^{(2n)} + \sum_{i=1}^{n} A_{2i-1}y^{(2i-1)}. \tag{1}$$

Following up this last observation, one finds that ay, $by''+b'y'$, $cy^{iv}+2c'y'''-c'''y'$ are self-adjoint and that $cy^{iv}+2c'y'''+by''+(b'-c''')y'+ay$ is the most general self-adjoint differential expression of the fourth order. It becomes reasonable to expect that for given $A(x)$ there is exactly one self-adjoint expression of

type (1). This expectation is correct but the existence does not seem to be obvious and the exact form of the expression does not seem to appear in the literature. To obtain the desired expression, we start with

THEOREM I. *Let* $\{x_{2r-1}\}$ *and* $\{y_{2r-1}\}$ *be the solution of the algebraic equations*

$$x_{2r-1} + \sum_{i=1}^{r} \binom{2r-1}{2i-1} x_{2i-1} = 1, \qquad \sum_{i=1}^{r} \binom{2r}{2i-1} y_{2i-1} = 1 \quad (r = 1, 2, \cdots).$$

Then $x_{2r-1} = y_{2r-1} = r^{-1}(2^{2r}-1)B_{2r}$; $B_2, B_4, \cdots,$ *are the Bernoulli numbers.** If we let $\tanh \frac{1}{2}x = \sum_{i=1}^{\infty} (c_{2i-1}x^{2i-1})/(2i-1)!$ we get

$$\sum_{r=1}^{\infty} \frac{x^{2r}}{(2r)!} = \cosh x - 1 = \tanh \tfrac{1}{2}x \sinh x$$

$$= \sum_{i=1}^{\infty} \frac{c_{2i-1}x^{2i-1}}{(2i-1)!} \sum_{s=0}^{\infty} \frac{x^{2s+1}}{(2s+1)!} = \sum_{i=1}^{\infty} \sum_{s=0}^{\infty} \frac{c_{2i-1}x^{2(i+s)}}{(2i-1)!(2s+1)!}$$

$$= \sum_{i=1}^{\infty} \sum_{r=1}^{\infty} \frac{c_{2i-1}x^{2r}}{(2i-1)!(2r-2i+1)!} = \sum_{r=1}^{\infty} x^{2r} \sum_{i=1}^{r} \frac{c_{2i-1}}{(2i-1)!(2r-2i+1)!}.$$

Equating coefficients, we get

$$\sum_{i=1}^{r} \frac{c_{2i-1}}{(2i-1)!(2r-2i+1)!} = \frac{1}{(2r)!}; \qquad \sum_{i=1}^{r} \binom{2r}{2i-1} c_{2i-1} = 1; \qquad y_{2r-1} = c_{2r-1}.$$

A similar procedure gives us

$$\sum_{r=1}^{\infty} \frac{x^{2r-1}}{(2r-1)!} = \sinh x = \tanh \tfrac{1}{2}x + \tanh \tfrac{1}{2}x \cosh x$$

$$= \sum_{i=1}^{\infty} \frac{c_{2i-1}x^{2i-1}}{(2i-1)!} + \sum_{i=1}^{\infty} \frac{c_{2i-1}x^{2i-1}}{(2i-1)!} \sum_{s=0}^{\infty} \frac{x^{2s}}{(2s)!}$$

$$= \sum_{r=1}^{\infty} \frac{c_{2r-1}x^{2r-1}}{(2r-1)!} + \sum_{r=1}^{\infty} x^{2r-1} \sum_{i=1}^{r} \frac{c_{2i-1}}{(2i-1)!(2r-2i)!};$$

$$1 = c_{2r-1} + \sum_{i=1}^{r} \binom{2r-1}{2i-1} c_{2i-1}; \qquad x_{2r-1} = c_{2r-1}.$$

Finally, the relation

$$\tanh \frac{1}{2}x = \sum_{i=1}^{\infty} \frac{(2^{2i}-1)B_{2i}x^{2i-1}}{(2i-1)!i}$$

produces the desired result.

* Professor E. J. McShane called my attention to the theorem: *Any result about the binomial coefficients is in the literature.* I cannot produce a suitable reference so I give a proof of this result.

THEOREM II. *The differential expression*

$$L(y) = Ay^{(2n)} + \sum_{i=0}^{n-1} \binom{2n}{2i+1} c_{2n-2i-1} A^{(2n-2i-1)} y^{(2i+1)}, \quad c_{2r-1} = r^{-1}(2^{2r}-1)B_{2r},$$

is self-adjoint.

To prove this theorem, we simplify the adjoint

$$M(y) = \sum_{s=0}^{2n} \binom{2n}{s} A^{(2n-s)} y^{(s)} - \sum_{i=0}^{n-1} \sum_{s=0}^{2i+1} \binom{2n}{2i+1} \binom{2i+1}{s} c_{2n-2i-1} A^{(2n-s)} y^{(s)}$$

by using

$$\sum_{s=0}^{2n} a_s = a_{2n} + \sum_{k=0}^{n-1} (a_{2k} + a_{2k+1}); \quad \binom{2n}{2i+1}\binom{2i+1}{s} = \binom{2n}{s}\binom{2n-s}{2n-2i-1};$$

$$\sum_{i=0}^{n-1} \sum_{s=0}^{2i+1} b_{is} = \sum_{k=0}^{n-1} \sum_{i=k}^{n-1} (b_{i,2k} + b_{i,2k+1}).$$

Thus

$$\begin{aligned} M(y) &= Ay^{(2n)} + \sum_{k=0}^{n-1} \binom{2n}{2k} A^{(2n-2k)} y^{(2k)} \left[1 - \sum_{i=k}^{n-1} \binom{2n-2k}{2n-2i-1} c_{2n-2i-1} \right] \\ &\quad + \sum_{k=0}^{n-1} \binom{2n}{2k+1} A^{(2n-2k-1)} y^{(2k+1)} \left\{ 1 - \sum_{i=k}^{n-1} \binom{2n-2k-1}{2n-2i-1} c_{2n-2i-1} \right\} \\ &= Ay^{(2n)} + \sum_{k=0}^{n-1} \binom{2n}{2k} A^{(2n-2k)} y^{(2k)} \left[1 - \sum_{r=1}^{n-k} \binom{2n-2k}{2r-1} c_{2r-1} \right] \\ &\quad + \sum_{k=0}^{n-1} \binom{2n}{2k+1} A^{(2n-2k-1)} y^{(2k+1)} \left\{ 1 - \sum_{r=1}^{n-k} \binom{2n-2k-1}{2r-1} c_{2r-1} \right]. \end{aligned}$$

Theorem I (with the r replaced by $n-k$) is applicable; the expression in brackets becomes zero, the expression in braces becomes $c_{2n-2k-1}$, and we have $M(y) = L(y)$.

Theorem II along with the remark (d) allows us to state that the most general self-adjoint differential expression of order $2n$ is

$$\sum_{s=0}^{n} A_s y^{(2s)} + \sum_{s=1}^{n} \sum_{k=0}^{s-1} \binom{2s}{2k+1} \frac{2^{2s-2k}-1}{s-k} B_{2s-2k} A_s^{(2s-2k-1)} y^{(2k+1)}.$$

Using $B_1 = -\frac{1}{2}$, $0 = B_3 = B_5 = \cdots$, this can also be written as

$$\sum_{s=0}^{n} \sum_{k=0}^{2s} (-1)^{k+1} \binom{2s}{k} \frac{2^{2s-k+1}-1}{2s-k+1} 2B_{2s-k+1} A_s^{(2s-k)} y^{(k)}.$$

THE NUMBER OF UNITARY DIVISORS OF AN INTEGER

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In this note n will represent a positive integral variable, and x a real variable, $x \geqq 2$. Let $\tau(n)$ denote the number of positive divisors of n. It is a classical elementary result of Dirichlet, dating from 1849, that

$$T(x) \equiv \sum_{n \leqq x} \tau(n) = x(\log x + 2\gamma - 1) + O(\sqrt{x}), \tag{1}$$

where γ is Euler's constant.

Suppose now that $t(n)$ represents the number of *unitary* divisors of n; that is, the number of (positive) divisors d of n such that d is relatively prime to the complementary divisor n/d. In 1874 Mertens proved that

$$T^*(x) \equiv \sum_{n \leqq x} t(n) = \frac{x}{\zeta(2)}\left(\log x + 2\gamma - 1 - \frac{2\zeta'(2)}{\zeta(2)}\right) + O(\sqrt{x}\log x), \tag{2}$$

where $\zeta(s)$ is the Riemann ζ-function and $\zeta'(s)$ its derivative.

Proofs of (1) have appeared in several texts in English, for example, LeVeque ([4], Th. 6.31). Although proofs of (2) have been reproduced in German texts (compare Landau [3], p. 666), none seems to be readily accessible in English. It is the purpose of the present note to correct this situation.

Mertens's proof of (2) is based on the Dirichlet estimate (1) for $T(x)$. In the more direct proof below we shall require only the much weaker estimate

$$T(x) = O(x \log x). \tag{3}$$

In place of the divisor function, our proof will be based mainly upon properties of totient functions.

Let $\phi(x, n)$ denote the number of positive integers $\leqq x$ which are prime to n, and place $\phi(n) = \phi(n, n)$. We note the following elementary results:

$$\phi(x, n) = \phi(n)(x/n) + O(\tau(n)), \tag{4}$$

uniformly in $x \geqq 1$;

$$\sum_{n \leqq x} \frac{\phi(n)}{n} = \frac{x}{\zeta(2)} + O(\log x); \tag{5}$$

$$\sum_{n \leqq x}^{*} \frac{\phi(n)}{n^2} = \frac{1}{\zeta(2)}\left(\log x - \frac{\zeta'(2)}{\zeta(2)} + \gamma\right) + O\left(\frac{\log x}{x}\right). \tag{6}$$

For a proof of (6), see [2], Lemma 2.4. The result (5) is well known; a generalization may be found in Wintner ([5], $(29)_4$, $f(n) = \phi(n)/n$). The relation (4) actually holds with $\tau(n)$ replaced by $t(n)$ and follows readily from the familiar evaluation of $\phi(x, n)$ in terms of the Möbius function ([1], Lemma 3.4; also *cf.* [6]).

Proof of (2). With (d, δ) denoting the greatest common divisor of d and δ, we have

$$T^*(x) = \sum_{n \le x} \sum_{d\delta=n}{}' 1 = \sum_{d\delta \le x}{}' 1,$$

where the prime indicates that in the summation $(d, \delta) = 1$. If $d\delta \leqq x$, then clearly d and δ cannot simultaneously assume values $> \sqrt{x}$. Hence

$$T^*(x) = 2 \sum_{d\delta \le x, d \le \sqrt{x}}{}' 1 - \sum_{d \le \sqrt{x}, \delta \le \sqrt{x}}{}' 1 = 2 \sum_{d \le \sqrt{x}} \phi(x/d, d) - \sum_{d \le \sqrt{x}} \phi(\sqrt{x}, d),$$

the prime again indicating that $(d, \delta) = 1$. Applying (4), $T^*(x)$ becomes

$$T^*(x) = 2x \sum_{n \le \sqrt{x}} \frac{\phi(n)}{n^2} - \sqrt{x} \sum_{n \le \sqrt{x}} \frac{\phi(n)}{n} + O(T\sqrt{x}). \tag{7}$$

Application to (7) of (3), (5), and (6) yields $(x \geqq 4)$

$$T^*(x) = 2x\left(\frac{1}{\zeta(2)}\left(\frac{1}{2}\log x - \frac{\zeta'(2)}{\zeta(2)} + \gamma\right) + O\left(\frac{\log x}{\sqrt{x}}\right)\right)$$
$$- \sqrt{x}\left(\frac{\sqrt{x}}{\zeta(2)} + O(\log x)\right) + O(\sqrt{x}\log x),$$

from which (2) results, on simplification. The extension of the result to the full range $x \geqq 2$ is immediate.

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ASYMPTOTICS I: A NOTE ON LAPLACE'S METHOD*

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Laplace's method has long been an important tool for examining the behavior of

$$I(x) = \int_a^b e^{-xh(t)} g(t)\,dt$$

as $x \to \infty$. This note is for the purpose of weakening the hypotheses under which an asymptotic expansion of $I(x)$ can be obtained. We treat the case where $[a, b]$ contains only one critical point (*i.e.*, minimum of h). For this case we have the best possible result in a certain sense: all we ask is that the integral exist for

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some x_0 and that the asymptotic behavior of h and g be known at the critical point, but we make no smoothness assumptions.

We make no attempt to survey the literature of this problem, but refer to Erdélyi (*Asymptotic Expansions*, New York, 1956) who treats the problem in Section 2.4. We are indebted to the referee for the remark that the possibility of removing the smoothness conditions was mentioned by Bekessy (Magyar Tud. Akad., vol. 2, 1957), and that a remark by Erdélyi at the end of Section 2.4 hints of this possibility.

For simplicity, and without loss of generality, we make the following normalizations: (1) the lower limit of integration is the critical point, (2) the lower limit of integration is 0, and (3) $f(0)=0$.

THEOREM. *Let h and g be measurable functions on $[0, b]$ which satisfy the following conditions*:

(1) *There is an x_0 for which $e^{-x_0 h}g \in L[0, b]$.*

(2) *For each a, $0<a<b$, there is an $A>0$ so that $h(t) \geqq A$ if $t \geqq a$.*

(3) $h(t)=t^{\nu}\sum_0^N a_k t^k+o(t^{N+\nu})$ *as* $t\to 0$, *where* $\nu>0$, $a_0>0$.

(4) $g(t)=t^{\lambda-1}\sum_0^N b_k t^k+o(t^{N+\lambda-1})$ *as* $t\to 0$, *where* $\lambda>0$ *and not all b's are zero.*

Then $I(x)=\int_a^b e^{-xh}g\,dt$ *exists for* $x \geqq x_0$ *and*

$$I(x) = \sum_0^N c_k x^{-(k+\lambda)/\nu} + o(x^{-(N+\lambda)/\nu}) \text{ as } x \to \infty,$$

where the c's are determined by the a's and b's.

Proof. $I(x)$ exists for $x \geqq x_0$: $e^{-x_0 h}g$ forms a dominating function since $h \geqq 0$. For any a, $0<a<b$ we have

$$I(x) = \int_0^a e^{-xh}g\,dt + \int_a^b e^{-xh}g\,dt = I_1(x) + I_2(x).$$

Now

$$|I_2(x)| = \left|\int_a^b e^{-xh}g\,dt\right| \leqq e^{-(x-x_0)A}\int_a^b e^{-x_0 h}|g|\,dt = \text{const.}\, e^{-Ax}.$$

For any given ϵ choose a so small that

$$\left|h(t) - t^{\nu}\sum_0^N a_k t^k\right| \leqq \epsilon t^{N+\nu} \text{ in } [0, a],$$

$$h_{\pm}(t) \equiv t^{\nu}\sum_0^N a_k t^k \pm \epsilon t^{N+\nu}$$

are both strictly increasing in $[0, a]$, and $g(t)$ is of one sign throughout $[0, a]$. This can be done since g will have the sign of the first nonvanishing b in its asymptotic development. Without loss of generality we may assume that $g(t)$ is nonnegative.

We next define I_+ and I_- by

$$I_+(x) = \int_0^a e^{-xh_+} g\, dt, \quad I_-(x) = \int_0^a e^{-xh_-} g\, dt.$$

Now h_+ and h_- both satisfy the conditions of Erdélyi's theorem. Thus

$$I_\pm(x) = \sum_0^{N-1} c_k x^{-(k+\lambda)/\nu} + c_N(\epsilon) x^{-(N+\lambda)/\nu} + o(x^{-(N+\lambda)/\nu}),$$

where it is easy to see from Erdélyi's proof that

(1) c_k is independent of ϵ for $k = 0, 1, \cdots, N-1$,

(2) $c_N(\epsilon)$ is linear in ϵ: $c_N(\epsilon) = c_N \mp \epsilon c_N'$,

(3) the o-term depends on ϵ.

Thus, since $I_+ \leqq I_1 \leqq I_-$ we easily get

$$\left| \left(I - \sum_0^N c_k x^{-(k+\lambda)/\nu} \right) x^{(N+\lambda)/\nu} \right| \leqq \epsilon c_N' + o(1) + O(e^{-Ax} \cdot x^{(N+\lambda)/\nu}).$$

Letting $x \to \infty$ and then $\epsilon \to 0$ we see that $I = \sum_0^N c_k x^{-(k+\lambda)/\nu} + o(x^{-(N+\lambda)/\nu})$.

The precise evaluation of the c's in terms of the a's and b's is described by Erdélyi. In particular we might mention that if b_n is the first nonvanishing b then c_n is the first nonvanishing c and then $c_n = a_0^{(n+\lambda)/\nu} \Gamma((n+\lambda)/\nu) b_n/\nu$. If all the b's vanish then these same calculations show that $I(x) = o(x^{-(N+\lambda)/\nu})$.

ON A PROBLEM OF L. FEJES-TÓTH*

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L. Fejes-Tóth has shown† that a point in 3-space may be "hidden" by six congruent spheres; more precisely, six congruent, nonintersecting closed spheres in E^3 may be placed around a point P (not belonging to any of the spheres) in such a way that each ray issuing from P intersects at least one of the spheres§. He also raised the question as to the least number of spheres for which such an arrangement is possible.

We shall show that five congruent, disjoint spheres may not "hide" a point, and prove the slightly stronger

THEOREM. *Let a point P and five congruent, open, disjoint spheres S_i, $1 \leqq i \leqq 5$, be given in E^3. If P belongs to none of the spheres S_i then there exists a ray R, issuing from P, such that $R \cap (\bigcup_{i=1}^5 \overline{S}_i)$ is either the point P, or is empty.*

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† Unpublished to the best of our knowledge. We are indebted to Professor V. L. Klee for this information.

§ It is not difficult to see that such configurations may be obtained by arbitrarily small, suitable transformations of the six spheres of radius 1 whose centers are (in cylindrical coordinates (r, ϕ, z)): $(0, 0, 1)$, $(0, 0, -1)$, $(\sqrt{3}, \phi_0, 0)$, $(\sqrt{3}, -\phi_0, 0)$, $(\sqrt{3}, 3\phi_0, 0)$, $(\sqrt{3}, -3\phi_0, 0)$ where $\phi_0 = \arccos \sqrt{\frac{2}{3}}$. The "hidden" point is $(0, 0, 0)$.

The proof of the theorem will follow from some simple lemmata.

Throughout this note we denote by S_i congruent, open, disjoint spheres, and by P a (fixed) point belonging to none of the spheres S_i. With any family of spheres S_i we associate the family of (open) spherical caps C_i on the surface of a fixed sphere S with center P, where C_i is obtained by projecting S_i onto Bd S by rays issuing at P. We denote by α_i the (angular) diameter of C_i, and by A_i the center of S_i.

LEMMA 1. *For any two spheres S_1 and S_2 we have* $\operatorname{diam}(C_1 \cap C_2) < \frac{1}{2}\pi$.

Proof. Since $\operatorname{diam}(C_1 \cap C_2)$ does not decrease if the centers A_i of the spheres S_i are moved toward P we may, without loss of generality, assume that P belongs to Bd S_1 and that the closed spheres $\overline{S}_1$ and $\overline{S}_2$ touch in a point. Let T be the plane passing through P and tangent to $\overline{S}_1$. If A_2 belongs to T or to that open half-space determined by T which contains A_1, an easy computation yields $\operatorname{diam}(C_1 \cap C_2) = \alpha_2 \leqq \arcsin \frac{2}{3}\sqrt{2} < \frac{1}{2}\pi$. On the other hand, if T separates A_1 from A_2 it follows by elementary trigonometry that the angle ϕ, spanned at P by $T \cap \overline{S}_2$, is given by

$$\phi = 2 \arcsin \frac{[(1 - \cos\vartheta)\cos\vartheta]^{1/2}}{\sin\vartheta},$$

where $\vartheta = \measuredangle PA_1A_2$. Since in this case $\operatorname{diam}(C_1 \cap C_2) \leqq \phi$, and $\phi < \frac{1}{2}\pi$ for $\vartheta > 0$, we have to settle only the case $\vartheta = 0$. But then $C_1 \cap C_2 = \varnothing$, *i.e.*, $\operatorname{diam}(C_1 \cap C_2) = 0$. This completes the proof of the lemma.

LEMMA 2. *If $\overline{C}_1 \cap \overline{C}_2 \cap \overline{C}_3 \neq \varnothing$ then* $\min(\alpha_1, \alpha_2, \alpha_3) \leqq \arcsin \frac{2}{3}\sqrt{2} < \frac{5}{12}\pi$.

Proof. From the assumption of the lemma it follows that there exists a ray R, issuing from P, such that $R \cap \overline{S}_i \neq \varnothing$ for $i = 1, 2, 3$. The centers A_1, A_2, A_3, of the spheres are contained in one of the closed half-spaces determined by the plane T passing through P and orthogonal to R. Let A_3 be not nearer to P than A_1 and A_2 (*i.e.*, $\alpha_3 \leqq \alpha_1, \alpha_2$). Since any motions of S_1 and S_2 (subject to the limitations $S_i \cap S_j = \varnothing$ for $i \neq j$, $\overline{S}_i \cap R \neq \varnothing$ for $i = 1, 2, 3$, and $\alpha_1, \alpha_2 \geqq \alpha_3$) do not change $\alpha_3 = \min \alpha_i$, it is easily seen that S_1 and S_2 may be moved (keeping S_3 fixed) in such a way that $A_1, A_2 \in T$ (and therefore R is their common tangent). But in this case a simple computation shows that $\alpha_3 \leqq \arcsin \frac{2}{3}\sqrt{2}$, which ends the proof of Lemma 2.

LEMMA 3. *For any three spheres S_i, $i = 1, 2, 3$, we have* $\min(\alpha_1, \alpha_2, \alpha_3) \leqq \frac{2}{3}\pi$.

Proof. Let T be a plane containing the centers A_i of the spheres S_i. (If the A_i are not collinear, T is unique.) If P is orthogonally projected onto T, the angles α_i increase. But if $P \in T$, then α_i obviously equals the angle spanned at P by the circle $T \cap \overline{S}_i$, and the assertion of the lemma is clearly true. Indeed, we have $\min \alpha_i < \frac{2}{3}\pi$ unless $\overline{S}_i \cap \overline{S}_j \neq \varnothing$ and P is the centroid of the (equilateral) triangle A_1, A_2, A_3, in which case $\alpha_i = \frac{2}{3}\pi$ for $i = 1, 2, 3$.

Proof of the theorem. Let us assume that there exists a configuration of five spheres S_i contradicting the theorem. If $P \in \overline{S}_1 \cap \overline{S}_2$ then $\overline{C}_3 \cap \overline{C}_4 \cap \overline{C}_5$ must cover a great circle on Bd S. But this is impossible since, by Lemma 2, $\alpha_i \leqq \arcsin \frac{2}{3}\sqrt{2} < \frac{1}{2}\pi$ for $i=3, 4, 5$. Therefore P belongs at most to one of the spheres $\overline{S}_i$. By the assumption made on S_i we have $\bigcup_{i=1}^{5} \overline{C}_i \supset \text{Bd } S$. We shall prove this to be impossible. Let $\alpha_1 \geqq \alpha_2 \geqq \alpha_i$ for $i=3, 4, 5$. Since in this case $\operatorname{diam}(\overline{C}_1 \cap \overline{C}_2) = \operatorname{diam}(C_1 \cap C_2) < \frac{1}{2}\pi$ by Lemma 1, the complement of $C_1 \cup C_2$, and thus also $\overline{C}_3 \cup \overline{C}_4 \cup \overline{C}_5$, contains an arc $> \frac{3}{2}\pi$ of a great circle on Bd S. Now, at least two of the three numbers α_3, α_4, α_5 are smaller than $\arcsin \frac{2}{3}\sqrt{2}$ (by Lemma 2) since obviously there exist points of Bd S covered by three of the sets $\overline{C}_i$ and not all such points are contained in one of the sets $\overline{C}_i$. Assume α_4, $\alpha_5 \leqq \arcsin \frac{2}{3}\sqrt{2}$; then $\alpha_3 > \frac{3}{2}\pi - 2 \arcsin \frac{2}{3}\sqrt{2} > \frac{2}{3}\pi$. But since $\alpha_1 \geqq \alpha_3$, $\alpha_2 \geqq \alpha_3$, we have reached a contradiction to Lemma 3. This completes the proof of the theorem.

Remark. It is easily seen that 4 spheres of different sizes may "hide" a point. On the other hand it seems probable that 4 spheres of two (possibly also three) different sizes may not "hide" a point.

THE FALSITY OF A CERTAIN ASYMPTOTIC RELATION

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On page 56 of the booklet* by Ernst Trost there is the statement that $p_1 \cdot p_2 \cdot \cdots \cdot p_{\pi(N)} \sim e^N$, which for $N = p_n$ implies

$$M_n = \prod_{p \leqq p_n} p \sim e^{p_n}. \tag{1}$$

This statement was suspect because its acceptance together with some newly derived results of F. B. Correia implied the truth of Cramér's conjecture which is $d_n = p_{n+1} - p_n = O(\log^2 p_n)$. The following theorem is therefore offered to show the falsity of Trost's statement.

THEOREM. *The relation* (1) *is false.*

Proof. If we suppose the contrary, then $M_n = e^{p_n}(1+o(1))$ and $\log M_n = p_n + \log(1+o(1)) = p_n + o(1)$. Thus

$$\log p_{n+1} = \log M_{n+1} - \log M_n = p_{n+1} - p_n + o(1) = d_n + o(1),$$

$$\frac{d_n}{\log p_n} = \frac{\log p_{n+1}}{\log p_n} + o\left(\frac{1}{\log p_n}\right) = \frac{\log p_{n+1}}{\log p_n} + o(1).$$

Since $\log p_{n+1} \sim \log p_n$ we have $(\log p_{n+1})/(\log p_n) = 1 + o(1)$ and $d_n/\log p_n = 1 + o(1) + o(1) = 1 + o(1)$, so that $\lim_{n\to\infty} d_n/\log p_n = 1$. This is a contradiction since it is known† that $\limsup_{n\to\infty} d_n/\log p_n = \infty$.

* Primzahlen, Bd. II, Basel-Stuttgart, 1953.

† Karl Prachar, Primzahlverteilung, Berlin-Gottingen-Heidelberg, 1957, p. 157.

CLASSROOM NOTES

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THE CENTROID IN ABSOLUTE GEOMETRY

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This paper deals with properties of the centroid of a tetrahedron that do not require any assumption about parallelism. We need only Hilbert's axioms of incidence, order, and congruence ([3], pp. 3–14).

Let us state specifically two theorems that are simple consequences of the axioms. (I) Every line segment has a midpoint ([3], p. 25). (II) The line joining the midpoints of two sides of a triangle is orthogonal to the perpendicular bisector of the third side ([1], Sec. 77; [3], p. 44). Furthermore, without quoting them verbally, we will make use of almost all the usual theorems on perpendicular lines and planes whose proof does not involve parallels ([2], pp. 182–183; [4], p. 43). We are now ready to prove the

THEOREM. *The lines joining the midpoints of opposite edges of a tetrahedron intersect in the centroid.*

For the proof let A, B, C, D be the vertices of a tetrahedron. Midpoints (I) are simply denoted by (AB), etc. Then the line through (AC), (BC) is perpendicular to the perpendicular bisector of edge AB in the plane ABC (II). A similar statement applies to the perpendicular bisector of AB in plane ABD. Clearly, line AB is perpendicular to the plane of the perpendicular bisectors mentioned and so are planes ABC and ABD. Hence line (AC), (BC) is orthogonal to the plane of the perpendicular bisectors and similarly line (AD), (BD). It follows that the two lines are coplanar. Hence lines (AC), (BD) and (AD), (BC) intersect in a point G. For reasons of analogy line (AB), (CD) passes through this point also. Now the planes $AD(BC)$; $BD(AC)$; $CD(AB)$ pass through point G and line DG. The intersections of these three planes and plane ABC are the medians $A(BC)$; $B(AC)$; $C(AB)$. Thus we see that the medians are concurrent at the point of intersection of line DG and plane ABC. This proves that G is the centroid of the tetrahedron and incidentally that the medians of a triangle intersect.

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ARITHMETIC, GEOMETRIC INEQUALITY

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The fact that, for $x_1, \cdots, x_n$ positive,

$$\sqrt[n]{(x_1 \cdots x_n)} \leqq \frac{x_1 + \cdots + x_n}{n}, \tag{1}$$

has many proofs abounding in the literature. We give a new variant which seems much more direct and which is just the thing for a class equipped with mathematical induction and little else.

(1) is trivial for $n=1$, now suppose it to hold for $n=k$ and assume $x_1x_2 \cdots x_{k+1}=1$, $x_i>0$. It is our burden to prove $x_1+\cdots+x_{k+1}\geqq k+1$.

Since $x_1x_2 \cdots x_{k+1}=1$ it follows that one of the x_i's is $\geqq 1$. We may take this to be x_{k+1} and obtain $x_1x_2 \cdots x_k=1/x_{k+1}$. By (1), which is assumed to hold for $n=k$, we obtain $x_1+x_2+\cdots+x_k\geqq k/\sqrt[k]{x_{k+1}}$ and so $x_1+x_2+\cdots+x_{k+1} \geqq k/\sqrt[k]{x_{k+1}}+x_{k+1}$. The proof will be complete, then, once we establish

LEMMA. *If* $x\geqq 1$ *then* $k/\sqrt[k]{x}+x\geqq k+1$.

(The fact is that, for $x\geqq 1$, $k/\sqrt[k]{x}+x$ is an increasing function, as is easily shown by differentiation, but if differentiation is outlawed then the following proof may be used.)

Proof. First recall that, for $a\geqq 0$, $(1+a)^{k+1}\geqq 1+(k+1)a$. This is easily established by induction or, for that matter, by the binomial theorem. Hence, with $a=\sqrt[k]{x}-1$, we obtain $x\cdot\sqrt[k]{x}\geqq(k+1)\sqrt[k]{x}-k$, or, transposing and dividing, $k/\sqrt[k]{x}+x\geqq k+1$.

That the equality holds in (1) only when $x_1=x_2=\cdots=x_n$, can also be obtained by a trivial modification of the above proof.

ON SIMPLE CONTINUED FRACTIONS

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In most texts on higher algebra the expression $p_nq_{n-r}-p_{n-r}q_n$ is evaluated in terms of a's for $r=1, 2, 3, 4$, where p_n/q_n is the nth approximant (*i.e.*, convergent) to the continued fraction:

$$a_1+\frac{1}{a_2+}\,\frac{1}{a_3+}\cdots\frac{1}{a_n+}\cdots. \tag{1}$$

The aim of this paper is to evaluate that expression for any positive integer $r<n$ in the case of two particular simple continued fractions.

The numerator p_n and denominator q_n of the nth approximant p_n/q_n to the simple continued fraction (1) are defined by the equations:

$$p_n = a_np_{n-1} + p_{n-2}, \qquad q_n = a_nq_{n-1} + q_{n-2}, \qquad (n > 2).$$

From this it follows that $p_{n-2}=p_n-a_np_{n-1}$, $q_{n-2}=q_n-a_nq_{n-1}$. Now letting $K_{n,s}=p_nq_s-p_sq_n$, we have

$$K_{n,s} = K_{n,s+2} - a_{s+2}K_{n,s+1}. \tag{2}$$

Putting $s=n-1, n-2, \cdots, n-r$ in succession in (2) and then adding these r relations we get

$$K_{n,n-r} = (K_{n,n-1} - K_{n,n-r+1}) - \sum_{i=1}^{r-1} a_{n+1-i}K_{n,n-i} \qquad (1 < r < n),$$

since $K_{n,n+1}=K_{n,n-1}$ and $K_{n,n}=0$.

Now consider the following simple continued fraction with one recurring element:

$$a + \frac{1}{a+}\,\frac{1}{a+}\cdots. \tag{3}$$

Knowing that $\binom{r}{m}=r!/m!(r-m)!$, and $\binom{r}{m}=0$, $m>r$, $m<0$; and $\binom{r}{m}=1$ for $m=0$, $r=m=0$; we claim that $K_{n,n-r}=K_{n,n-r+2}-a_{n-r+2}K_{n,n-r+1}$ satisfies the relation

$$K_{n,n-r} = (-)^{r-1} \sum_{p=0}^{[\frac{1}{2}(r-1)]} \binom{r-1-p}{p} a^{r-1-2p}K_{n,n-1} \qquad (1 < r < n),$$

which can be restated as follows:

$$\begin{aligned} K_{n,n-2r} &= -\sum_{p=0}^{r-1}\binom{2r-1-p}{p} a^{2r-1-2p}K_{n,n-1}, \\ K_{n,n-(2r+1)} &= \sum_{p=0}^{r}\binom{2r-p}{p} a^{2r-2p}K_{n,n-1}. \end{aligned} \tag{4}$$

We prove (4) by induction on r. It is easily seen that (4) is true for $r=1$. Suppose it is true for $r=R$. Then

$$\begin{aligned} K_{n,n-2(R+1)} &= K_{n,n-2R} - aK_{n,n-(2R+1)} \\ &= -\left[\sum_{p=0}^{R-1}\binom{2R-1-p}{p} a^{2R-1-2p} + \sum_{p=0}^{R}\binom{2R-p}{p} a^{2R+1-2p}\right]K_{n,n-1}. \end{aligned}$$

But

$$\sum_{p=0}^{R-1}\binom{2R-1-p}{p} a^{2R-1-2p} = \sum_{q=1}^{R}\binom{2R-q}{q-1} a^{2R+1-2q} = \sum_{q=0}^{R}\binom{2R-q}{q-1} a^{2R+1-2q}.$$

Therefore

$$K_{n,n-2(R+1)} = -\left[\sum_{p=0}^{R}\left\{\binom{2R-p}{p-1} + \binom{2R-p}{p}\right\} a^{2R+1-2p}\right]K_{n,n-1}$$

$$= -\sum_{p=0}^{R}\binom{2R+1-p}{p} a^{2R+1-2p}K_{n,n-1}.$$

Next

$$K_{n,n-[2(R+1)+1]} = K_{n,n-(2R+1)} - aK_{n,n-2(R+1)}$$

$$= \left[\sum_{p=0}^{R}\binom{2R-p}{p} a^{2R-2p} + \sum_{p=0}^{R}\binom{2R+1-p}{p} a^{2R+2-2p}\right] K_{n,n-1}$$

$$= \left[\sum_{q=1}^{R+1}\binom{2R+1-q}{q-1} a^{2R+2-2q} + \sum_{p=0}^{R}\binom{2R+1-p}{p} a^{2R+2-2p}\right] K_{n,n-1}$$

$$= \left[\sum_{q=0}^{R+1}\binom{2R+1-q}{q-1} a^{2R+2-2q} + \sum_{p=0}^{R+1}\binom{2R+1-p}{p} a^{2R+2-2p}\right] K_{n,n-1}$$

$$= \sum_{p=0}^{R+1}\binom{2R+2-p}{p} a^{2R+2-2p}K_{n,n-1}.$$

Thus (4) is true for $r=R+1$ whenever it is true for $r=R$. This completes the proof.

We thus get the

THEOREM. *If p_n/q_n and p_{n-r}/q_{n-r} $(r=2, 3, \cdots, n-1)$ are the nth and $(n-r)$th approximants to the continued fraction* (3), *then*

$$p_n/q_n - p_{n-r}/q_{n-r} = \frac{(-)^{n+r-1}}{q_n \cdot q_{n-r}} \cdot \left\{\sum_{p=0}^{[\frac{1}{2}(r-1)]}\binom{r-1-p}{p} a^{r-1-2p}\right\},$$

since $K_{n,n-1} = (-)^n$.

For the simple continued fraction with two recurring elements:

$$a + \frac{1}{b+}\,\frac{1}{a+}\,\frac{1}{b+}\cdots$$

$K_{n,n-(2r+1)}$ and $K_{n,n-2r}$ satisfy the relations

$$K_{n,n-(2r+1)} = \sum_{p=0}^{r}\binom{2r-p}{p}(ab)^{r-p}K_{n,n-1} \quad (n \text{ odd or even}),$$

$$K_{n,n-2r} = -\sum_{p=0}^{r-1}\binom{2r-1-p}{p} a^{r-p}b^{r-1-p}K_{n,n-1} \quad (n \text{ odd}),$$

$$K_{n,n-2r} = -\sum_{p=0}^{r-1}\binom{2r-1-p}{p} b^{r-p}a^{r-1-p}K_{n,n-1} \quad (n \text{ even}).$$

The proof by induction on r is omitted.

A METHOD OF CALCULATING THE CHARACTERISTIC EQUATION OF A MATRIX

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The purpose of this note is to make readily available to students and teachers of matrix algebra an interesting, and relatively simple method of finding the characteristic equation and then the inverse of a matrix. This method is briefly mentioned in references [1] and [5], but the proof is omitted and does not seem to be readily available in the literature. The total number of arithmetic operations required by this method is distinctly greater than for many routines designed purely for matrix inversion. However, the method is straightforward, finite and easily checked. I have found that programming this procedure for both small and large digital computers is relatively easy. The greatest advantage of this method is the bulk of useful information obtained.

DEFINITION. *The characteristic polynomial* $f(\epsilon)$ *of the square matrix* A *is defined by* $f(\epsilon)=\det[\epsilon I-A]$. *The characteristic equation of* A *is* $f(\epsilon)=0$, *and the roots of this equation are the characteristic roots, or eigenvalues, of* A.

It is a simple matter to show that $f(\epsilon)$ is monic, of degree equal to the order of the matrix, and that the constant term is $(-1)^n \det[A]$, n being the order of A [6]. The remaining coefficients in the characteristic equation are not so readily obtained, and it is this topic that I shall pursue further. Once the characteristic equation has been obtained, it may be used to obtain the inverse of the matrix. The theoretical justification here is the classic Cayley-Hamilton theorem: *A matrix satisfies, in the matrix sense, its own characteristic equation* [6].

DEFINITION. *The trace of a matrix* A, $\operatorname{tr}(A)$, *is the sum of the diagonal elements of* A.

The result in question, Theorem 5 below, follows from the following familiar theorems. Consider

$$f(\epsilon) = \det[\epsilon I - A] = \prod(\epsilon - \epsilon_i) = \epsilon^n + a_1\epsilon^{n-1} + \cdots + a_{n-1}\epsilon + a_n.$$

THEOREM 1. [6]. $\sum\epsilon_i = -a_1 = \operatorname{tr}(A)$.

THEOREM 2. [3]. $a_j = (-1)^j \sum\epsilon_1 \cdots \epsilon_j$.

THEOREM 3 [2]. (Newton's formula for symmetric polynomials.) *If* S_k *denotes the symmetric polynomial* $\sum(\epsilon_j)^k$, *and if* p_j *denotes the symmetric polynomial* $\sum\epsilon_1 \cdots \epsilon_j$, *then, for* $r \leqq n-1$

$$S_r - p_1S_{r-1} + p_2S_{r-2} - p_3S_{r-3} + \cdots + (-1)^r(r)p_r = 0.$$

THEOREM 4 [6]. *If* ϵ_i *is a characteristic root of* A, *then* ϵ_i^k *is a characteristic root of* A^k *and* $\operatorname{tr}(A^k) = \sum(\epsilon_i)^k = S_k$.

THEOREM 5. *The coefficients of* $f(\epsilon)$ *may be evaluated by the recursion formula*

$$a_k = [-1/k][a_{k-1}S_1 + a_{k-2}S_2 + \cdots + S_k]; \qquad S_k = \operatorname{tr}(A^k).$$

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Proof. By Theorem 2, $a_j=(-1)^j\sum\epsilon_1\cdots\epsilon_j=(-1)^jp_j$, and by Theorem 4, $S_j=\operatorname{tr}(A^j)=\sum(\epsilon_i)^j$. Using Newton's formula we have,

$$S_j - p_1S_{j-1} + p_2S_{j-2} + \cdots + (-1)^{j-1}p_{j-1}S_1 + (-1)^jjp_j = 0.$$

Since $a_j=(-1)^jp_j$, this is equivalent to $S_j+a_1S_{j-1}+a_2S_{j-2}+\cdots+a_{j-1}S_1+ja_j=0$, from which it follows that $a_j=[-1/j][a_{j-1}S_1+a_{j-2}S_2+\cdots+S_j]$.

Example.

$$A=\begin{bmatrix}1&0&2&-1&4\\5&3&-1&0&1\\8&5&-3&1&4\\6&2&0&0&1\\0&1&4&2&0\end{bmatrix},\qquad A^2=\begin{bmatrix}11&12&12&9&11\\12&5&14&-4&19\\15&6&36&-3&26\\16&7&14&-4&26\\49&27&-13&4&19\end{bmatrix},$$

$$A^3=\begin{bmatrix}221&125&18&23&113\\125&96&53&40&105\\315&218&20&73&207\\139&109&87&50&123\\104&43&186&-24&175\end{bmatrix},\qquad A^4=\begin{bmatrix}1128&624&715&23&1104\\1269&738&415&138&848\\2003&1107&1180&119&1631\\1680&985&400&194&1063\\1663&1186&307&432&1179\end{bmatrix},$$

$$A^5=\begin{bmatrix}10106&6597&3903&1795&8019\\9107&5413&3947&842&7612\\17692&11090&5883&2439&13958\\10969&6406&5427&846&9499\\12641&7136&5935&1002&9498\end{bmatrix}.$$

We then find $S_1=1$, $S_2=67$, $S_3=562$, $S_4=4419$, and $S_5=31746$. The recursion formula of Theorem 5 then yields $a_1=-1$, $a_2=-33$, $a_3=-154$, $a_4=-373$, and $a_5=382=(-1)^5\det[A]$. The characteristic polynomial is

$$\epsilon^5 - \epsilon^4 - 33\epsilon^3 - 154\epsilon^2 - 373\epsilon + 382.$$

From the Cayley-Hamilton theorem we have

$$A^{-1} = -1/382[-373I - 154A - 33A^2 - A^3 + A^4],$$

$$A^{-1}=\frac{-1}{382}\begin{bmatrix}17&103&-7&-143&12\\-22&-358&54&230&-38\\-39&-79&61&-9&-50\\89&337&-149&-97&-72\\-58&98&-66&16&4\end{bmatrix}.$$

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COMBINATORIAL MISCELLANEA

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In combinatorial analysis there is often a need to determine the number of combinations of a set of elements with or without regard to the arrangement and with or without repetition. From a more general point of view, this leads to the evaluation of the cardinal of certain sets of functions. Moreover, if we specialize the derived formulae, we receive some interesting relations between binomial coefficients. Recall that a function f, defined in A with values in B, is said to be injective, if $x \neq y$ implies $f(x) \neq f(y)$; to be surjective, if $f(A) = B$; to be bijective, if it is both injective and surjective. (See [1], Ch. II, and [2].)

PROPOSITION 1. *Let A and B be finite sets having p and q elements, respectively. Denote by F_{pq}* (resp., I_{pq}, S_{pq}, B_{pq}) *the number of the functions (resp., injections, surjections, bijections) defined in A with values in B. The following formulae hold (with $0^0 = 1$):*

$$F_{pq} = q^p, \qquad I_{pq} = \binom{q}{p} \cdot p!,$$

$$S_{pq} = \sum_{i=0}^{q} (-1)^i \binom{q}{i} \cdot (q - i)^p, \qquad B_{pq} = p!\delta_{pq} \qquad \left(\delta_{pq} = \begin{cases} 1, & \text{if } p = q \\ 0, & \text{if } p \neq q \end{cases}\right).$$

The proofs of the first, second and fourth formulae can be found, say, in [1], Ch. III (see also [3]), while the third formula appears as an exercise. We indicate the proof. Obviously, $F_{pq} = \sum_{i=0}^{q} \binom{q}{i} S_{pi}$ is true due to the fact that $\binom{q}{i}$ is the number of the parts with i elements of a set with q elements. The third formula is now an immediate consequence of the

LEMMA. *Let f and g be two maps of the set of natural numbers N into N. If, for all $n \in N$, the equation $f(n) = \sum_{i=0}^{n} \binom{n}{i} g(i)$ holds, then the same is true for the equation $g(n) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} f(n-i)$.*

Indeed,

$$\sum_i (-1)^i \binom{n}{i} f(n - i) = \sum_{i,j} (-1)^i \binom{n}{i} \binom{n - i}{j} g(j)$$

$$= \sum_{i,j} (-1)^i \binom{n - j}{i} \binom{n}{j} g(j) = g(n).$$

COROLLARY. *Let p and q be natural numbers. Then*

a) $$\sum_{i=0}^{p} (-1)^i \binom{p}{i} (p - i)^p = p!;$$

b) $$p < q \text{ implies } \sum_{i=0}^{q} (-1)^i \binom{q}{i} (q - i)^p = 0.$$

PROPOSITION 2. *Let A and B be totally ordered finite sets having p and q elements, respectively. Denote by F_{pq}^{+} (resp., I_{pq}^{+}, S_{pq}^{+}, B_{pq}^{+}) the number of the increasing functions (resp., injections, surjections, bijections) defined in A with values in B. The following formulae hold (with $\binom{-1}{0}=1$):*

$$F_{pq}^{+}=\binom{p+q-1}{p},\quad I_{pq}^{+}=\binom{q}{p},\quad S_{pq}^{+}=\sum_{i=0}^{q}(-1)^{i}\binom{q}{i}\binom{p+q-i-1}{p},$$

$$B_{pq}^{+}=\delta_{pq}.$$

The proofs of the second and fourth formulae can be found in [1], the first formula is an immediate consequence of the second one, and the proof of the third formula is similar to that one given in Proposition 1.

COROLLARY. *Let p and q be natural numbers. Then*

a) $$\sum_{i=0}^{p}(-1)^{i}\binom{p}{i}\binom{2p-i-1}{p}=1.$$

b) $$p<q \text{ implies } \sum_{i=0}^{q}(-1)^{i}\binom{q}{i}\binom{p+q-i-1}{p}=0.$$

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EXISTENCE OF A MAXIMUM OF A CONTINUOUS FUNCTION

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The existence theorem for a maximum of a function continuous on a closed interval is often stated in terms of the boundedness of the function and the attainment of its least upper bound. Perhaps because of this many of the standard proofs of the theorem depend essentially on the existence and properties of the least upper bound of a bounded function. This note contains a version of the familiar bisection proof in which the existence of a maximum is proved directly and the idea of a least upper bound is not mentioned. The proof is based on the fact that a bounded increasing sequence has a finite limit.

There seem to the author to be two advantages in using such a proof in an elementary course in analysis. Firstly, the required result is obtained in one stage and not two. Secondly, such a proof makes possible the postponement of the definition of a least upper bound until after the basic theorems on continuity and the derivative (and even the Riemann integral if it is defined as a limit) have been proved. Of course, the standard "compactness" proof based on the Heine-Borel theorem has both these properties and is much more general, but it cannot be called elementary.

THEOREM. *If the real-valued function f is continuous on the closed interval $[a, b]$, there is a number $\xi \in [a, b]$ such that $f(x) \leqq f(\xi)$ for every $x \in [a, b]$.*

Proof. Let $a_0 = a$ and $b_0 = b$. We define a_{n+1} and b_{n+1} inductively in terms of a_n and b_n. Let $c_n = \frac{1}{2}(a_n + b_n)$. If there is a number $u \in [a_n, c_n]$ such that $f(v) \leqq f(u)$ for every $v \in [c_n, b_n]$, let $a_{n+1} = a_n$ and $b_{n+1} = c_n$. If not (that is, if for each $u \in [a_n, c_n]$ there is a number $v \in [c_n, b_n]$ such that $f(v) > f(u)$), let $a_{n+1} = c_n$ and $b_{n+1} = b_n$. In either case

(i) $a_n \leqq a_{n+1} \leqq b_{n+1} \leqq b_n$;
(ii) $b_{n+1} - a_{n+1} = \frac{1}{2}(b_n - a_n)$;
(iii) if $x \in [a_n, b_n]$, there is a number $y \in [a_{n+1}, b_{n+1}]$ such that $f(x) \leqq f(y)$.

This inductive process defines a sequence of intervals $[a_n, b_n]$ $(n = 0, 1, 2, \cdots)$ with the properties:

(i)′ $a_m \leqq a_n \leqq b_m \leqq b_0$ if $n \geqq m \geqq 0$;
(ii)′ $b_n - a_n \to 0$ as $n \to \infty$;
(iii)′ if $x \in [a, b]$ and N is a nonnegative integer, there is a number $y \in [a_N, b_N]$ such that $f(x) \leqq f(y)$.

By (i)′, the sequence a_n tends to a finite limit, ξ say, as $n \to \infty$, and $\xi \in [a_m, b_m]$ for $m = 0, 1, 2, \cdots$. In particular, $\xi \in [a, b]$. We shall prove that $f(x) \leqq f(\xi)$ for every $x \in [a, b]$.

Let $x^* \in [a, b]$. Choose $t > f(\xi)$. Since f is continuous on $[a, b]$ and $\xi \in [a, b]$, there is a positive number δ such that $f(y) < t$ whenever $y \in [a, b]$ and $|y - \xi| < \delta$. By (ii)′ we can choose N so that $b_N - a_N < \delta$, and then, by (iii)′, we can choose $y^* \in [a_N, b_N]$ so that $f(x^*) \leqq f(y^*)$. Since $\xi \in [a_N, b_N]$, we have $|y^* - \xi| \leqq b_N - a_N < \delta$. Also $y^* \in [a, b]$. Hence, $f(y^*) < t$ and, consequently, $f(x^*) < t$. It follows that $f(x^*) \leqq f(\xi)$, for if this were not so we could take $t = f(x^*)$ and obtain a contradiction.

We remark, in conclusion, that the above proof applies without essential change to an upper-semicontinuous mapping f of $[a, b]$ into any linearly ordered set T. (One extra step is required to deal with the case when there is no $t \in T$ such that $t > f(\xi)$; in this case we cannot have $f(x^*) > f(\xi)$ since $f(x^*) \in T$, and so we must have $f(x^*) \leqq f(\xi)$.) Also, it is not difficult to replace $[a, b]$ by any closed bounded subset of the real line (or of real Euclidean n-space). But for such results it is more natural to use the Heine-Borel theorem.

A POWER SERIES DEVELOPMENT OF THE CONVOLUTION THEOREM

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Let the function $F(t)$ have the Laplace transform

$$L[F(t)] = f(s) = \int_0^\infty e^{-st} F(t)\,dt$$

and recall the well-known formula

$$(1) \qquad L\left[\int_0^t \int_0^t \cdots \int_0^t F(t)dt^{n+1}\right] = f(s)/s^{n+1},\ n \text{ a nonnegative integer.}$$

This formula is really just a special case of the convolution theorem. For, if the identity*

$$\int_0^t \cdots \int_0^t F(t)dt^{n+1} = \int_0^t \frac{F(z)(t-z)^n}{n!}\,dz,$$

is used to replace the multiple integral in (1) by a single integral, there follows

$$(2) \qquad L\left[\int_0^t F(z)(t-z)^n dz\right] = f(s)(n!/s^{n+1}) = L[F(t)]\cdot L[t^n],$$

which we recognize as the convolution theorem as it pertains to $F(t)$ and nonnegative powers of t.

In order to extend the theorem to $F(t)$ and the polynomial

$$P(t) = \sum_{k=0}^{n} a_k t^k, \qquad a_1, a_2, \cdots, a_n \text{ constants,}$$

we observe, in view of (2) and the linearity of the operator L, that

$$L\left[\int_0^t F(z)P(t-z)dz\right] = \sum_{k=0}^{n} a_k L\left[\int_0^t F(z)(t-z)^k dz\right]$$

$$= f(s)\sum_{k=0}^{n} a_k k!/s^{k+1} = L[F(t)]L[P(t)].$$

Now let $G(t)$ be any function which admits of a Maclaurin series representation $G(t) = \sum_{k=0}^{\infty} G^{(k)}(0)t^k/k!$, $t>0$. Then, proceeding formally, we obtain

$$L\left[\int_0^t F(z)G(t-z)dz\right] = L\left[\int_0^t F(z)\sum_{k=0}^{\infty} G^{(k)}(0)(t-z)^k/k!\,dz\right]$$

$$= \sum_{k=0}^{\infty} G^{(k)}(0)/k! \int_0^t F(z)(t-z)^k dz$$

$$= f(s)\sum_{k=0}^{\infty} G^{(k)}(0)/s^{k+1} = L[F(t)]L[G(t)].$$

Having arrived at the convolution theorem by means of these elementary considerations, it is natural to suspect that the theorem is true under less stringent conditions on $G(t)$. Thus, the stage is set for a more general, and more rigorous treatment of the theorem using classical methods.

* F. B. Hildebrand, Methods of Applied Mathematics, Englewood Cliffs, N. J., 1952, p. 384.

CONICS ON A SPHERE

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The following results were obtained after a discussion in a freshman class which I taught at Purdue University during the fall of 1958. I have never seen this topic treated elsewhere, but Professor Philip Dwinger has suggested that it very likely occurs in some European textbooks.

1. Basic definitions. If p is a point on a sphere, then $\pi(p)$ denotes the diametrically opposite point, *i.e.*, $\pi(p)$ is the projection through the center of the sphere of the point p. If p_1 and p_2 are two points on the sphere which are not diametrically opposite, then $[p_1, p_2]$ denotes the shorter arc joining p_1 and p_2 of the great circle through p_1 and p_2. We assume $[p_1, p_2]$ is topologically closed, *i.e.*, p_1 and p_2 are in $[p_1, p_2]$. The length of this arc is the *distance*, $d(p_1, p_2)$, *between* p_1 *and* p_2. For diametrically opposite points we define $d(p, \pi(p)) = \frac{1}{2}c$ where c is the circumference of the sphere. For any three points p_1, p_2, p_3 on the sphere we have:

$$d(p_1, p_2) \leqq \tfrac{1}{2}c, \tag{1.1}$$

$$d(p_1, p_3) \leqq d(p_1, p_2) + d(p_2, p_3), \tag{1.2}$$

$$d(p_1, p_2) + d(p_1, \pi(p_2)) = \tfrac{1}{2}c. \tag{1.3}$$

Using this metric, the distance between two sets may be defined in the usual way. In particular, if L is a great circle and p is a point of the sphere not on L, then $d(p, L)$, the *distance between* p *and* L, is either $\frac{1}{4}c$ or is the length of the shorter arc joining p and L of the great circle through p orthogonal to L.

2. The ellipse. Analogous to the definition usually given for the ellipse in the plane, we define the *ellipse* on a sphere (of circumference c) with points f_1 and f_2 (not necessarily distinct) as *foci* and *semimajor axis of length* a to be the set of all points p on the sphere such that

$$d(p, f_1) + d(p, f_2) = 2a. \tag{2.1}$$

By using (1.3), (2.1) is easily shown to be equivalent to

$$d(p, \pi(f_1)) + d(p, \pi(f_2)) = 2(\tfrac{1}{2}c - a). \tag{2.2}$$

Hence:

(2.3) THEOREM. *The ellipse with foci* f_1 *and* f_2 *and semimajor axis of length* a *is identical to the ellipse with foci* $\pi(f_1)$ *and* $\pi(f_2)$ *and semimajor axis of length* $\frac{1}{2}c - a$.

The triangular inequality (1.2) and (2.1) imply $d(f_1, f_2) \leqq 2a$. Likewise (2.2) implies $d(f_1, f_2) = d(\pi(f_1), \pi(f_2)) \leqq 2(\frac{1}{2}c - a)$. Hence if the graph of (2.1) is not empty, then

$$d(f_1, f_2) \leqq 2a \leqq c - d(f_1, f_2). \tag{2.4}$$

Conversely, if (2.4) is satisfied, then the ellipse, the graph of (2.1), is not empty. Under the restriction (2.4) some special cases of the ellipse are:

(2.5) If $f_1 = f_2$, then the ellipse is a circle for $0 < 2a < c$ and is a point for $2a = 0$ or $2a = c$.

(2.6) If $f_1 = \pi(f_2)$, then $2a = \frac{1}{2}c$ and (2.1) is satisfied by all points of the sphere.

(2.7) If $f_1 \neq f_2$, $f_1 \neq \pi(f_2)$, and $d(f_1, f_2) = 2a$, the ellipse is the arc $[f_1, f_2]$. If $2a = c - d(f_1, f_2)$, the graph of (2.1) is the arc $[\pi(f_1), \pi(f_2)]$. If $2a = \frac{1}{2}c$, then the ellipse is a great circle.

3. The parabola. The *parabola* on a sphere with the great circle L as *directrix* and the point f as *focus* is the set of all points p on the sphere such that

$$d(p, L) = d(p, f). \tag{3.1}$$

We observe that if f is on L, then the parabola is that arc of a great circle which joins the "north and south pole" (*i.e.*, the extremities of the diameter perpendicular to the plane of L) and passes through the point f. All other cases are included in the following:

(3.2) THEOREM. *The parabola with the "equator" E as directrix and focus f in the "northern hemisphere" is identical to the ellipse with the point f and the "north pole" n_0 as foci and with semimajor axis of length $\frac{1}{8}c$.*

This theorem is derived from (3.1) by use of the identity

$$d(p, n_0) + d(p, E) = \tfrac{1}{4}c, \tag{3.3}$$

which holds for all points p in the "northern hemisphere." We note that if $f = n_0$ the parabola is a circle consisting of all points of latitude 45° north.

4. The hyperbola. The *hyperbola* on a sphere (of circumference c) with points f_1 and f_2 (not necessarily distinct) as *foci* and *semitransverse axis of length* a is the set of all points on the sphere such that

$$d(p, f_1) - d(p, f_2) = 2a \tag{4.1}$$

or

$$d(p, f_2) - d(p, f_1) = 2a. \tag{4.2}$$

By virtue of (1.3), (4.1) is equivalent to

$$d(p, f_2) + d(p, \pi(f_1)) = 2(\tfrac{1}{4}c - a) \tag{4.3}$$

and (4.2) is equivalent to

$$d(p, f_1) + d(p, \pi(f_2)) = 2(\tfrac{1}{4}c - a). \tag{4.4}$$

Hence:

(4.5) THEOREM. *The hyperbola with foci f_1 and f_2 and semitransverse axis of length a is the union of two ellipses, each with semimajor axis of length $\frac{1}{2}c-a$. The foci of one ellipse are the points f_1 and $\pi(f_2)$. The foci of the other are f_2 and $\pi(f_1)$.*

Consequently:

(4.6) THEOREM. *The hyperbola with foci f_1 and f_2 and semitransverse axis of length a is identical to the hyperbola with foci $\pi(f_1)$ and $\pi(f_2)$ and semitransverse axis of length a.*

We also observe that one elliptic component of a hyperbola is the image under the projection π of the other.

From (1.3), (2.4), and (4.5) we have: *the hyperbola is not empty if and only if*

$$-d(f_1, f_2) \leqq 2a \leqq d(f_1, f_2), \tag{4.7}$$

There are several special cases:

(4.8) If $f_1=f_2$, then $2a=0$ and the hyperbola is the whole sphere.

(4.9) If $f_1\neq f_2$ and $2a=0$, then the hyperbola is a single great circle.

(4.10) If $f_2=\pi(f_1)$ and $2a\neq 0$ then the hyperbola is a pair of circles the planes of which are perpendicular to the diameter joining f_1 and f_2.

(4.11) If $2a=d(f_1, f_2)$ or $-d(f_1, f_2)$, then the hyperbola is the union of the two arcs $[f_1, \pi(f_2)]$ and $[f_2, \pi(f_1)]$.

ON THE DEFINITION OF THE RIEMANN INTEGRAL*

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The now classic approach to the theory of the Riemann integral of a bounded function f on a closed interval $[a, b]$, $a<b$, is based on the concept of a partition. A *partition* P of $[a, b]$ is any finite ordered set of numbers $\{x_0, x_1, \cdots, x_p\}$ satisfying

$$a = x_0 < x_1 < \cdots < x_p = b.$$

The intervals $[x_0, x_1]$, $[x_1, x_2]$, $\cdots$, $[x_{p-1}, x_p]$ are the *subdivisions* of P; the *norm* of P is the length of its longest subdivision.

Using P, we now form a *Riemann sum*

$$\sum_{k=1}^{p} f(\xi_k)(x_k - x_{k-1}), \qquad x_k \text{ in } P, \tag{1}$$

where ξ_k is *any* number in the kth subdivision of P. We shall say that P *underlies* the sum (1). (Note that in general may different sums may have the same

* The writing of this paper was facilitated by the Carnegie Foundation, under a grant to the Illinois Institute of Technology.

underlying partition.) What happens now has been aptly characterized by E. J. McShane ([2], p. 6):

" . . . We take some kind of limit of this sum. The question is, what kind of limit. Sometimes one hears 'the limit as the number of subdivisions tends to infinity,' with some qualifying remark about the lengths of the subdivisions. But this sum is not a single-valued function of the number of subdivisions. Some kind of extension of the concept of limit is called for. One alternative is to look sternly at the student and say 'That's perfectly clear, isn't it?' . . . A better way is to widen the theory to cover multiple-valued functions, treating the value of a sum as a multiple-valued function of the length of the largest subinterval . . . . A third way . . . is to use the Moore-Smith limit . . . ."

The purpose of this note is to discuss a fourth way, in which the integral is defined, rigorously and simply, as an ordinary limit of an ordinary sequence of numbers. Now "sequence" definitions of integrals are not new. Both N. Wiener ([4], p. 366) and K. Menger ([3], p. 141) have defined

$$\int_a^b f = \lim_{n\to\infty} \sum_{k=1}^{n} f\left(a + (k-\alpha)\frac{b-a}{n}\right)\frac{b-a}{n}, \tag{2}$$

with $\alpha=0$, $\alpha=\frac{1}{2}$, respectively, whenever the limit exists. This definition yields the Riemann integral of all Riemann-integrable functions, and has the advantages of simplicity, adaptability to manual or machine computations, and ease of proof of certain properties of the integral, *e.g.*, linearity. Against these advantages, however, are certain drawbacks. For example, the function g defined by setting

$$g(x) = \begin{cases} 1, & x \text{ rational}, \\ 0, & x \text{ irrational}, \end{cases}$$

is integrable under (2), (though not Riemann-integrable). However, we have $\int_0^1 g = 1$, but $\int_0^c g = 0$ for *any* irrational c; and c may be arbitrarily close to 1. Thus under (2), the integral is not a continuous function of the interval of integration.

This fact is related to the fact that under (2) the basic property of *interval-additivity*, expressed by

$$\int_a^b f + \int_b^c f = \int_a^c f, \tag{3}$$

will not be generally valid. Taking g to be the function defined above and c to be any irrational number, we have, for example: $\int_0^1 g + \int_1^c g = 1 + 0 \neq 0 = \int_0^c g$. Furthermore, as pointed out by Menger ([3], p. 142), even for functions for which (3) is valid, this validity is not easy to derive from (2).

These drawbacks can be overcome by the following procedure: For any integer n, let $P_n(a, b)$ consist of the numbers a, b, and all numbers between a and b of the form $m/2^n$, m an integer. If there are $p+1$ numbers (including a and b) in $P_n(a, b)$, let these be denoted in their order of magnitude by x_0,

$x_1, \cdots, x_p$. Thus $P_n(a, b)$ is a partition of $[a, b]$, but of a special type. We now define Darboux-type upper and lower sums

$$ {}_n\overline{S}_a^b f = \sum_{k=1}^{p} M_k(x_k - x_{k-1}), \qquad {}_n\underline{S}_a^b f = \sum_{k=1}^{p} m_k(x_k - x_{k-1}), $$

where M_k and m_k denote, respectively, the least upper bound and the greatest lower bound of f on the kth subdivision of $P_n(a, b)$. We can show without difficulty that the ${}_n\overline{S}$'s form a bounded nonincreasing sequence and the ${}_n\underline{S}$'s a bounded nondecreasing sequence. Since *any* bounded monotonic real sequence has a real limit, it follows that

$$ \lim_{n\to\infty} {}_n\overline{S}_a^b f \quad \text{and} \quad \lim_{n\to\infty} {}_n\underline{S}_a^b f $$

both exist. We write

$$ \lim_{n\to\infty} {}_n\overline{S}_a^b f = \overline{\int_a^b} f, \qquad \lim_{n\to\infty} {}_n\underline{S}_a^b f = \underline{\int_a^b} f, \tag{4} $$

and call these limits, respectively, the *upper* and the *lower integral* of f on $[a, b]$. If the upper and lower integrals are equal, we call their common value the *integral* of f on $[a, b]$ and denote it by the usual symbol. Accordingly, whenever the integral exists, we have

$$ \int_a^b f = \lim_{n\to\infty} {}_n\overline{S}_a^b f = \lim_{n\to\infty} {}_n\underline{S}_a^b f. \tag{5} $$

The upper and lower integrals of (4) coincide with the Darboux-Riemann upper and lower integrals; while the integral of (5) coincides with the ordinary Riemann integral. To see this, we need only observe, firstly, that the partitions P_n underlying the sums ${}_n\overline{S}$ and ${}_n\underline{S}$ have norms which are not greater than 2^{-n} (and consequently approach zero as n tends to infinity). We then appeal to the following theorems:* Any sequence of upper (lower) sums converges to the upper (lower) Darboux-Riemann integral, provided only that the underlying partitions have norms which approach zero; if the upper and lower Darboux-Riemann integrals are equal, their common value is the Riemann integral, and conversely, if the Riemann integral exists, the upper and lower Darboux-Riemann integrals are equal.

However, it should be noted that *the entire theory of integration can be based on* (4) *and* (5) *without any direct appeal to the customary theory.* For example, the fundamental property (3) can be derived from (4) and (5) on the basis of the following simple observation: By virtue of the definitions of P_n, ${}_n\overline{S}$ and ${}_n\underline{S}$, we have, for any n, and for $a \leqq b \leqq c$,

* Proofs may be found in, *e.g.*, Graves [1], pp. 86–87.

$$(6) \qquad {}_n\overline{S}_a^b f + {}_n\overline{S}_b^c f = {}_n\overline{S}_a^c f + \alpha_n, \qquad {}_n\underline{S}_a^b f + {}_n\underline{S}_b^c f = {}_n\underline{S}_a^c f + \beta_n,$$

where α_n and β_n satisfy $|\alpha_n| \leqq 2^{-n} \sup |f|$, $|\beta_n| \leqq 2^{-n} \sup |f|$. Letting n tend to infinity in (6), we therefore obtain

$$\overline{\int}_a^b f + \overline{\int}_b^c f = \overline{\int}_a^c f, \qquad \underline{\int}_a^b f + \underline{\int}_b^c f = \underline{\int}_a^c f,$$

from which (3) follows as a corollary.

It is likewise not hard to prove the *existence* of the integral for continuous functions (if the concept of uniform continuity is available); it is easier still, indeed almost trivial, to demonstrate this existence for *monotonic* functions. Finally, it may be noted that, without unduly great effort, it is possible to show that for functions integrable under (4) and (5), the partitions P_n can be replaced by arbitrary partition schemes, and the sums ${}_n\overline{S}$, ${}_n\underline{S}$ by general Darboux-Riemann sums, or by ordinary Riemann sums. In this way, if desired, the classic definition of the Riemann integral can be recovered. In particular, the partitions P_n can be modified by replacing the factor 2^{-n}, which occurs in their definition, by the factors 3^{-n}, or 10^{-n}, or indeed by k^{-n}, where k is any integer $\geqq 1$.

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A SYSTEM ISOMORPHIC TO THE REALS

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In certain constructive approaches to the real number system, the nonnegative (or, strictly, unsigned) reals are encountered prior to the introduction of the (signed) reals—positive, zero, and negative. Various technical devices are available to effect the extension from the nonnegative reals to the reals. The reals can, for example, be introduced as equivalence classes of couples (ordered pairs) of nonnegative reals where two such couples, (u, v) and (r, s), are defined to be equivalent if and only if $u+s=v+r$. The traditional algebraic approach postulates the existence of additive inverses and introduces the negative reals as "tagged" positive reals, the minus sign playing a dual role as an operator (for subtraction) and as a tag for the negative reals.

In this paper we proceed heuristically and informally from the unsigned reals in decimal form $N \cdot a_1 a_2 a_3 \cdots$ (N integral and the a_i's digits from 0 to 9) to a

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novel system exhibiting the formal properties of the signed reals. Our method of development initially will be one of blind faith and unabashed juggling of symbols (in the tradition of the eighteenth century analysts), following where our intuitions lead. Only later will a definition introduce meaning and show that we have not been led to absurdity.

Consider now the equation $x+3=0$. In a system of tagged reals $x=-3$ is a solution. If we remove the restriction, in our decimal expansions, that N be an integer—*i.e.*, that our decimals terminate on the left—there is another (formal) solution, namely, $x=\cdots 9997.$, as suggested by the formal calculation:

$$\begin{array}{r} \cdots 9997. \\ +\ 3 \\ \hline \cdots 0000. \end{array}$$

providing we take $\cdots 0000.$ in our new system as an expansion of zero. If further we take $\cdots 0001.=1$, $\cdots 0002.=2$, etc., we have by similar experimentation the following equivalences among the elements of our new system and the system of tagged reals

$$\begin{aligned} \cdots 9999. &= -1 \\ \cdots 9998. &= -2 \\ \cdots 9997. &= -3 \\ &\ \ \vdots \qquad\ \vdots \\ \cdots 9990. &= -10 \\ &\ \ \vdots \qquad\ \vdots \\ \cdots 9900. &= -100 \\ &\ \ \vdots \qquad\ \vdots \end{aligned}$$

As another example, a formal solution to the equation $x+5=10x$, namely $x=\cdots 4445.$, suggests the equivalence $\cdots 4445.=\frac{5}{9}$, since by formal calculation we have

$$\begin{array}{r} \cdots 4445. \\ +\ 5 \\ \hline \cdots 44450. \end{array} \qquad \text{and} \qquad \begin{array}{r} \cdots 4445. \\ \times\ 10 \\ \hline \cdots 44450. \end{array}$$

The same equivalence is suggested by the calculations:

$$\begin{array}{r} \cdots 4445. \\ \times\ 9. \\ \hline \cdots 0005. \end{array} \qquad 9\overline{)\cdots 0005.}\ \cdots 4445.$$

and $\cdots 4445.=1+(\cdots 444.)=1+\frac{4}{9}(\cdots 999.)=1+\frac{4}{9}(-1)=\frac{5}{9}$.

If we consider symbols nonterminating in either direction—right or left—from the decimal point we have, for example, $\cdots 999.999 \cdots = (\cdots 999.) + (.999 \cdots) = -1+1=0$ and thus $\cdots 999.999 \cdots$ reveals itself as another representation of zero. Representations in our system are thus seen to be not unique. Indeed we have by similar considerations

$$0 = \cdots 111 \cdot 111 \cdots = \cdots 222 \cdot 222 \cdots = \cdots 121212 \cdot 121212 \cdots$$
$$= \cdots xxx \cdot xxx \cdots \text{ (where x is any digit or block of digits).}$$

Indeed any real number has an infinitude of representations as the following suggest:

$$\tfrac{1}{2} = \cdots 000.5000 \cdots = \cdots 000.4999 \cdots = \cdots 111.6111 \cdots$$
$$= \cdots 222.7222 \cdots = \cdots 5556.0555 \cdots = \cdots 6667.1666 \cdots$$
$$= \cdots 101010.601010 \cdots = \cdots 121212.621212 \cdots$$
$$\tfrac{1}{7} = \cdots 142857142857143.000 \cdots = \cdots 000.142857142857 \cdots$$
$$= \cdots 1428571428571.571428571428 \cdots$$
$$\sqrt{2} = \cdots 0001.4142 \cdots = \cdots 1112.5253 \cdots = \cdots 2223.6364 \cdots$$
$$-\sqrt{2} = \cdots 9998.5857 \cdots = \cdots 8887 \cdot 4746 \cdots$$

(where the representations of $\sqrt{2}$ and $-\sqrt{2}$ are meant to be periodic to the left but not to the right).

Examples could be multiplied ad nauseam. We have displayed a number sufficient to motivate the following:

DEFINITION. *A double-decimal is a symbol composed of a decimal point preceded and followed by an infinite array of digits, the array to the left of the decimal point being periodic from some position to the left on.*

A double-decimal is thus seen to be of the form

$$\cdots \overline{c_n c_{n-1} \cdots c_2 c_1} \overline{c_n c_{n-1} \cdots c_2 c_1} b_m b_{m-1} \cdots b_1 b_0 . a_1 a_2 a_3 \cdots$$

where the a's, b's, and c's are digits, 0 to 9, and $\overline{c_n c_{n-1} \cdots c_2 c_1}$ is the part of the number repeating to the left. Thus $\cdots 000.101001000100001 \cdots$ (where the number of 0's between two 1's increases by one at each stage) is a double-decimal whereas $\cdots 100001000100101.000 \cdots$ is not.

We define the *value* of the double-decimal given above to be

$$\frac{1}{1-10^n}(10^{m+1}c_1 + 10^{m+2}c_2 + \cdots + 10^{m+n}c_n)$$
$$+ b_0 + 10b_1 + \cdots + 10^m b_m + \sum_{i=1}^{\infty} \frac{a_i}{10^i}.$$

The double-decimal is said to *represent* its value. Definitions of equality,

the arithmetic operations, and order can be readily supplied (see [3]). For example, two double-decimals are to be *equal* if their values are equal, *i.e.*, if their difference is some expansion of zero.

We conclude with the following observations:

(1) Of course there is nothing special about the radix ten. Any other base could be used.

(2) Every real number has an infinitude of double-decimal representations. Every double-decimal represents a unique real number.

(3) In view of (2), the reals can be defined as equivalence classes of double-decimals, where equivalence should be so defined as to constitute equality of value. The arithmetic operations are readily defined along well-known lines [3]. In the resulting system the reals (or their double-decimal representations) are not tagged and the minus sign reverts to its aboriginal role as operator. Of course the double-decimals can be tagged if it is desirable to do so.

(4) Considerations of technical efficacy would mitigate against the use of double-decimals in computations. For example, order loses the easy perceptibility of ordinary decimal expansions. Thus $\cdots 999.000 \cdots < \cdots 8889.000 \cdots$. Indeed the former number is negative (-1) whereas the latter is positive $(+\frac{1}{9})$.

(5) The process of "rounding off" can be carried out on the right as usual but not on the left. $\cdots 333.333000 \cdots$ is a fair approximation to $\cdots 333.333 \cdots$ but $\cdots 000333.333 \cdots$ is poor. And the more 3's we take on the left (short of all of them) the poorer it becomes!

(6) Notations of course have such meaning as we choose to give them (within limits imposed by requirements of consistency and adequacy). We have assigned no meaning to decimal arrays which are not periodic to the left.

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REMAINDER FORMULAE IN TAYLOR'S THEOREM

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It is not uncommon in an elementary calculus course to develop Taylor's formula with a remainder described on the one hand in integral form and on the other by use of an intermediate value of a derivative of the function being represented. Thus if we write

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + R_n,$$

we have

$$R_n = \int_a^x \frac{(s-t)^n}{n!} f^{(n+1)}(t)dt \tag{1}$$

or

$$R_n = \frac{(x-a)^{n+1}}{(n+1)!} f^{n+1}(X), \tag{2}$$

where $(X-a)(X-x)<0$. The first form is often arrived at by a succession of integrations by parts; the second by an application of Rolle's theorem to a suitable function.

The comparison of these two representations of the remainder provides an opportunity for introducing by example the generalized mean-value theorem for integrals (to use the terminology of Courant [1]), but we wish to show that it can also be used in an elementary course to motivate a consideration of distinctions between types of discontinuities.

For convenience (1) will be called the integral form of R_n, (2) Lagrange's form. Further we shall assume $a<x$; the necessary changes in the discussion will be obvious in the alternative case.

In an elementary treatment, it is usual to assume in arriving at the integral form that $f^{(n+1)}$ is continuous in the closed interval from a to x. At the same time, the generalized mean-value theorem for integrals, *viz.*,

$$\int_a^x p(t)\phi(t)dt = \phi(X)\int_a^x p(t)dt, \qquad (a < X < x), \tag{3}$$

is also usually proven under the assumptions: (a) $p(t)\geqq 0$, (b) p and ϕ continuous in the closed interval from a to x. Then one establishes directly Lagrange's remainder from the integral form. But in applying Rolle's theorem, Lagrange's form of the remainder is obtained under a weaker assumption on $f^{(n+1)}$—namely its existence in the open interval from a to x. This suggests that Taylor's formula with Lagrange's remainder applies to a wider class of functions.

However, Taylor's formula with the integral remainder could be established by integration by parts assuming, say, only Riemann integrability for $f^{(n+1)}$. This puts that form of Taylor's theorem in a little more favorable light; but the question now arises as to whether, if we wish, we can establish Lagrange's remainder by an integral mean-value theorem. To do this we extend the generalized mean-value theorem for integrals.

We shall say ϕ assumes intermediate values over an interval from a to b if for each pair x_0, x_1 of numbers selected from this interval and for any number y between $\phi(x_0)$ and $\phi(x_1)$, there is an x between x_0 and x_1 such that $\phi(x)=y$. This was, implicitly, the definition of continuity in less sophisticated times. It is easy to prove in the usual way that if in the closed interval from a to x: (i) $p(t)$

$\geqq 0$, (ii) ϕ assumes intermediate values, and (iii) the indicated integrals in (3) exist, then (3) is true.

Using this form of the mean-value theorem, Lagrange's remainder can be established as a consequence of the integral remainder under the assumptions that $f^{(n+1)}$ satisfies the conditions (ii) and (iii) just stated. But the interesting point which can be made is that (ii) is *automatically* satisfied when ϕ is a derivative, *e.g.*, $f^{(n+1)}$. That is, a derivative, even though discontinuous, must assume intermediate values, or put another way, a derivative cannot have jump discontinuities, (*cf.* [2]). To summarize, Taylor's theorem with integral remainder is valid if $f^{(n+1)}$ is, say, Riemann integrable; with Lagrange's remainder it is valid under the weaker condition that $f^{(n+1)}$ exist in the open interval from a to x.

The pedagogical point of the discussion is this. It is often observed that the integral remainder is better suited to some purposes and this furnishes an obvious motivation to the above extension of its range of validity. At the same time, it is an opportunity to show reasons for introducing finer distinctions in discussions of discontinuities. And finally there is perhaps some attraction in the device of displaying an apparent discrepancy of results followed by at least a partial resolution of the discrepancy.

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FEUERBACH'S THEOREM: A NEW PROOF

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In 1822 Feuerbach published at Nürnberg a little book of 62 pages entitled "Eigenschaften einiger merkwürdigen Punkte des geradlinigen Dreiecks und weiterer durch sie bestimmten Linien und Figuren." The British Museum has a copy of this rare book (which has not been translated into English). In it the author sets forth many properties of a triangle and its inscribed and escribed circles. Using the values he has found (Fig. 1) for OI, IH and NH in the formula $OI^2+IH^2=2NI^2+2NH^2$, it follows that $NI=\frac{1}{2}R-r$, which shows that the Nine-point Circle touches the inscribed circle—his famous theorem—extended easily to the escribed circles.

Since 1822 many new proofs have been provided. In 1842 Stubbs and Ingram, of Dublin, gave to the world the fruitful method of Inversion, which yielded simpler proofs than Feuerbach's. Properties of tangents and of conics leading to interesting generalizations have been applied suggestively (See, *e.g.*, Baker, *Introduction to Plane Geometry*, Appendix to Ch. XII). The simplest proof is one given in McClelland's *Geometry of the Circle* page 225, and in Lachlan's *Modern Pure Geometry* page 74.

As there are so many proofs—now I can reckon nine!—I was naturally surprised to be able to evolve a new proof which I give herewith.

R=rad. circum. centre O. r=rad. in-circ. centre I. H=orthocentre. K, G mid. pts. HE, HD, and lie on $N.P.C.$ $KG=R$=diam. $N.P.C.$ perp. to FP. GFK =right angle. N=centre $N.P.C.$ QX=diam. in-circle perp. to FP. XL is perp. to GK. $LM=XQ=2r$. FG, Simson line of D, is perp. to CD at S, and bisects DH at G. $DI.IC=DB.IC=2Rr$. Angles marked α are equal.

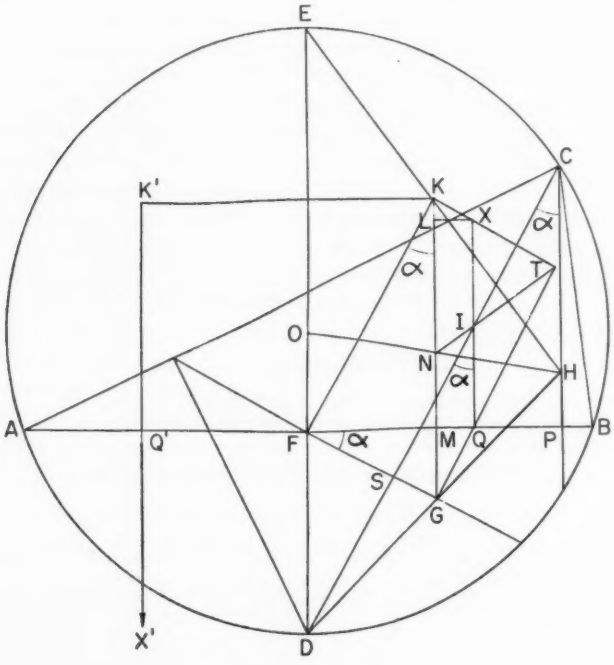

FIG. 1

To prove that GQ and KX, meeting at T, are at right angles. For then circles on diameters KG and QX, ($N.P.C.$ and in-circle) touch at T. $FQ=DI \sin\alpha$; $QP=IC \sin\alpha$; therefore $FQ.QP=DI.IC \sin^2\alpha=2Rr \sin^2\alpha$. But $R \sin^2\alpha = GK \sin^2\alpha=FG \sin\alpha=GM$. Hence $FQ.QP=2r.GM=LM.GM$. $FQ.QP+MQ^2 = MP^2 = MF^2 = KM.GM = (KL + LM)GM = LM.GM + KL.GM = FQ.QP + KL.GM$. Therefore $MQ^2=KL.GM=LX^2$.

Hence $(LX)/(KL)=(GM)/(MQ)$. Therefore, KX and GQ meet at right angles, and therefore the circles touch at T.

Mutatis mutandis it can be shown that GQ' and KX' which join ends of diameters of $N.P.C.$ and escribed circle cross at right angles. In this case $2R.r_3 \sin^2\alpha = Q'P.FQ$. $FQ = FQ'$. $Q'X' = 2r_3$. Hence easily $(X'K')/(K'K) = (Q'M)/(GM)$ and the result follows.

MATHEMATICAL EDUCATION NOTES

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TEACHING OF MATHEMATICS IN SWITZERLAND

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In the last years, my teaching activity was almost entirely devoted to additional training of teachers of mathematics and science. Participants in my classes often asked me questions about European education, and so I gave a few talks, outside class time, about this subject with some aspects of which I am well acquainted from personal experience. I wanted to use a few months' trip to Switzerland to gather additional information, especially about recent developments. I am greatly indebted to the Executive Committee of the Mathematical Association of America and to the U.S. Commission on Mathematical Instruction which designated me their official representative to visit Swiss secondary schools to gain an up-to-date picture of current trends in mathematical education in Switzerland.

The present report summarizes my impressions. It is so written that most of it can be submitted to any group of high school teachers or interested parents and can be read by people who have no time or patience to absorb many details or many unfamiliar terms.

Summary. In such complex human matters as the one before us, simplifications and generalizations are never very safe. Yet one cannot summarize without simplifications and generalizations. I am keenly conscious of all this when, in comparing the Swiss schools *with the average American high school*, I find the following four points the most significant:

1. *Division into specialized branches.* After primary school, that is, on the average from the seventh grade on, the school is divided into several specialized branches, and the curriculum within each branch leaves no choice, or little choice, to the student.

2. *Teacher training* emphasizes mastery of the subject-matter much more, practice teaching more, and general didactical lectures much less.

3. *Performance.* In those branches of the Swiss schools that prepare for university studies the programs in mathematics (and also in science and languages) go considerably further.

4. *Change.* No major change has taken place in the recent past, and no major change is likely to take place in the near future, although there is some gradual change, some discussion, and some experimentation.

The foregoing refers to mathematics as taught in an average Swiss school compared with an average U.S. high school for boys and girls of the same age. Yet, in fact, points 1, 2 and 3 (not point 4) are approximately correct to a much

further extent: the teaching of mathematics, science and languages in Central European schools shows, on the whole, the same contrast with the American high schools. By the way, all this is no news; it can be seen by any not quite casual and somewhat open-minded observer, it has been stated time and again —what is needed here is not so much fact-finding but rather fact-facing.

The following four sections illustrate the four points by adding appropriate details. An appendix says a few words about the personal experience of the author on which his opinions are based.

1. Division of the school into specialized branches. All the children (the future medical doctor, the future salesman, the future housewife, and so on) go to the same primary school, but eventually they must study different things and so their studies must be separated from a certain point onward: the question is when and how. The basic difference between the European and U.S. school systems concerns this *When* and *How*.

The annexed diagram "The Zurich School System" represents the system of public schools in the Canton of Zurich. (There is some simplification; more complex and less typical details are neglected.) I believe that it is a fairly representative example; any school system in Central Europe would show the same kind of contrast with American high schools.

The diagram shows clearly enough, I hope, how the school splits successively into several branches; it shows the number of years the students have to spend in each branch, and the students' age; it shows moreover (sketchily) the subjects covered by the mathematical instruction in each branch. It emphasizes the grades 7 through 13; it gives less space to the first six grades, says nothing about the university level. I wish to explain the abbreviations and add a few remarks. (I add, in parentheses, the original German terms and the corresponding terms used in the French-speaking Swiss cantons.)

PRIMARY: Primary school (Primarschule, école primaire) is for all children; it lasts six years. The separation into several branches begins after it, with the seventh grade.

PREP. A, B, C: Preparatory schools; they prepare the pupils especially for university studies.

PREP. A: Preparatory school of type A (Literargymnasium, gymnase littéraire) with Latin, Greek, and one living foreign language (French). Mathematics is taught in all grades through 6½ years.

PREP. B: Preparatory school of type B (Realgymnasium; gymnase) with Latin (no Greek) and two living foreign languages (usually French and English). The program in mathematics is the same as for type A.

PREP. C: Preparatory school of type C (Oberrealschule, gymnase scientifique). No Latin, two living foreign languages, more mathematics and science than in types A and B. It prepares principally for the study of Science and Engineering on university level.

SECONDARY: Secondary school (Sekundarschule, école secondaire). It prepares principally for professions which need no university studies. It is entered by the greatest fraction of students (about 60%) and has more resemblance to the American junior high school than other branches of the Swiss school system. Mathematics is taught in each grade.

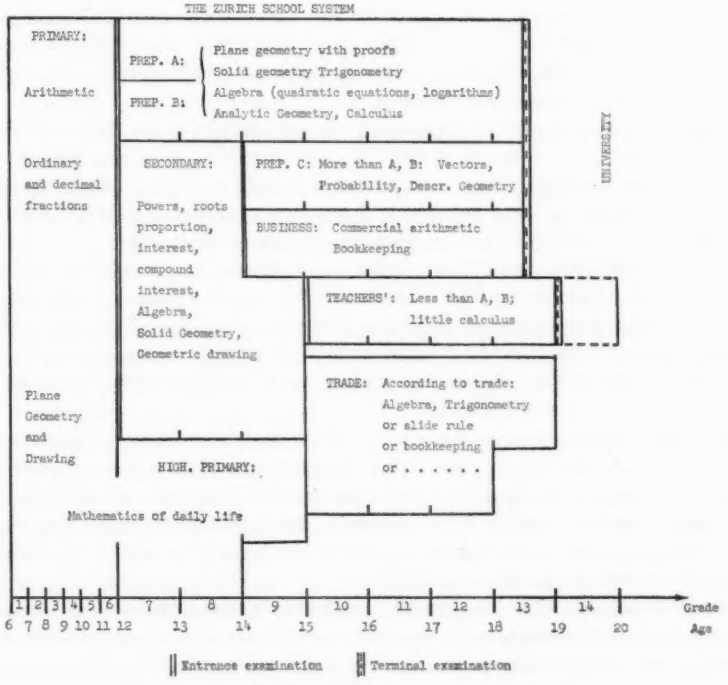
HIGH. PRIMARY: Higher primary school (Oberschule, école primaire supérieure). Still less theoretical than the secondary school.

TRADE: Trade school (Gewerbeschule, école professionelle). It has several branches, accord-

ing to the different professions, of varying duration and intensity. It is an obligatory part-time school (8 to 10 hours weekly) for "apprentices" who are trained in shops, workshops, offices or factories (they work also part time and receive some salary which increases as their training advances). Mathematical instruction varies greatly: quite strong for some trades, very little for some others.

BUSINESS: Business school (Handelsschule, école de Commerce).

TEACHERS': Teachers' preparatory school (Unterseminar, école normale). (See Sect. 2.)


THE ZURICH SCHOOL SYSTEM
PRIMARY:
Arithmetic
Ordinary and decimal fractions
Plane Geometry and Drawing
PREP. A:
PREP. B:
Plane geometry with proofs
Solid geometry Trigonometry
Algebra (quadratic equations, logarithms)
Analytic Geometry, Calculus
SECONDARY:
Powers, roots proportion, interest, compound interest, Algebra, Solid Geometry, Geometric drawing
HIGH. PRIMARY:
Mathematics of daily life
PREP. C: More than A, B: Vectors, Probability, Descr. Geometry
BUSINESS: Commercial arithmetic Bookkeeping
TEACHERS': Less than A, B; little calculus
TRADE: According to trade: Algebra, Trigonometry or slide rule or bookkeeping or . . . . . .
UNIVERSITY
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 Grade
6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 Age
Entrance examination
Terminal examination

Entrance examinations are marked in the diagram by double vertical line. For example, a pupil who has completed the primary school and wishes to enter the preparatory school of type A has to pass such an examination; he can enter the higher primary school without such an examination.

Terminal examinations which entitle the student to enter the university, or some branches of it (Maturitätsprüfung, maturité) are marked in the diagram by a triple vertical line (the middle line is dotted).

There is *full-time obligatory schooling* for 9 years to the age of 15 (in some cases only 8 years). Moreover, there is additional *part-time* obligatory schooling to the age 18; apprentices fulfill this obligation by attending trade school.

2. Teacher training.

Primary school. There is a special preparatory school (with two living foreign languages, no Latin) for prospective teachers of the primary and secondary schools (see TEACHERS' in diagram). In view of the future profession of the pupils, the program of this preparatory school contains subjects which are not obligatory in other preparatory schools (music, art drawing, manual skills, technique of speech) and teaches other subjects to a lesser extent. Mathematics is taught in all grades (5 hours per week in grade 10, four hours in the following grades): Algebra to quadratic equations and logarithms (also slide rule), plane geometry, solid geometry, some descriptive geometry (acceptable representation of solids on the blackboard), analytic geometry, differential calculus up to easy problems in maxima and minima.

Only the last year of this preparatory school brings some distinctively professional instruction (4 hours weekly in one semester, 3 in the other) to the prospective teachers and this instruction is mostly practical: visits to schools and practice teaching with following discussion; general didactical ideas presented in class are closely tied up with the practical exercises.

The TEACHERS' preparatory school is followed by the *one year teachers' training school on university level* (Oberseminar) which, again, is predominantly practical: lectures on general didactics, history of pedagogy, psychology and child psychology take weekly 5 hours out of 34 in one semester and 8½ hours out of 32½ in the other. The special didactics of various subjects are strongly practical; exercises are emphasized.

The main emphasis is on *practice teaching*: each candidate teaches 3 hours per week through the whole year under the guidance of an experienced teacher. Moreover, between the two semesters, he teaches 3 weeks full time in a city school and another 3 weeks in a country school. Having passed the training school, the candidate must teach successfully at least 40 weeks in the following two years to receive his final certificate.

Secondary schools. The prospective teacher in a secondary school must have the primary teacher's final certificate and two years practice in a primary school, after which he receives *two years training at the university.* The program of this training contains didactical lectures, but the emphasis is on the mastery of the subject.* There are six branches of science teaching, divided into two groups: (1) Mathematics, Physics, Chemistry;(2) Botany, Zoology, Geography. Any secondary teacher of science must take two subjects from group (1) and two subjects from group (2), do practice teaching in both groups and, in his final examination, teach a class for one hour in group (1) before the examiners, and another class in group (2). One of the branches chosen by any science teacher must be mathematics and the minimum requirement is a course in calculus.

Preparatory schools. A major fraction of the mathematics teachers for pre-

* For instance, language teachers must have lived in a place where the language is spoken at least 6 months, and speak it fluently.

paratory schools in all Swiss cantons receive their training at the Swiss Federal Institute of Technology (E.T.H., Eidgenössische Technische Hochschule, Ecole Polytechnique Fédérale) in Zurich (a high class Technical University).

The program for the final certificate (mathematician's diploma) demands solid knowledge both in classical and in modern branches of mathematics; there are also courses and examinations in mechanics and in experimental and theoretical physics. An extensive paper on a special mathematical subject, for which three months' work is allowed, is also part of the final examination. In short, a mathematician's diploma at the E.T.H. is roughly equivalent to a strong M.S. degree.

About 250 semester-hours' work is needed for the mathematican's diploma;† only 7 additional semester-hours are needed for the additional pedagogical examination which is also required for teaching in preparatory schools. These are distributed as follows:

General didactics, 3 semester hours: most of the time is spent on discussing the established school system, its purpose, its practical aspects.

Topics from school-mathematics, 2 semester-hours: discussion of the most important topics which should be viewed differently at the university and in the preparatory school.

Practice teaching, 2 semester-hours: either all candidates listen to a class conducted by an experienced teacher, or one candidate teaches and the others observe. Each hour of practice teaching is followed by a discussion (particularly useful if the instructor is experienced).

In the final examination, the candidate must teach twice, one hour each time, in presence of the examiners.

Before the present teacher shortage (which began in Switzerland relatively late, two or three years ago) the (unwritten) requirements for a preparatory school teacher were even higher: a Ph.D. degree in the better-paying city schools.

The teacher in a preparatory school has some advantages over the teacher in a secondary school: less teaching load, more salary, a little more security, but the differences are not so great: the salaries are in the proportion 5 to 4, or thereabout. Yet the preparatory school teacher can do research and may advance to a university position not infrequently. (A personal note: among my former students who wrote their Ph.D. dissertation at the E.T.H. under my guidance, three are now professors of the E.T.H.; all three taught several years in preparatory schools).

3. Performance. It is easy to compare programs, but it is not easy to judge the performance. For instance, there is a subject which is treated in Switzerland even in secondary schools, but which is conspicuously (and for the outsider, sur-

† A student takes around thirty hours per week, but several hours are devoted to practice sessions or to laboratory work. For instance, freshman calculus is taught 9 hours weekly: 6 hours lectures, 3 hours practice sessions. As a result there is relatively less homework than in an American university.

prisingly) absent from the program of the average American high school: divisibility and the decomposition into prime factors. Yet to compare in performance the primary schools on both sides of the Atlantic, or the Swiss secondary schools with the junior high schools, is not so easy. At any rate, I am not in a position to undertake such a comparison.

There is, however, a point where the comparison is possible: the young people who passed the terminal examination (Reifeprüfung, Maturitätsprüfung, maturité) of a Swiss (or German, or Austrian, . . .) preparatory school, are about two years ahead of those boys and girls who, having finished high school, enter the university: they are about the same age and attended school the same number of years, but the boy with the "Maturitätsprüfung" can be admitted to an American university as a junior* whereas the boy from the high school will be a freshman. As far as I can judge on the basis of my extensive experience on both sides of the Atlantic, this difference of two years persists: the boy who gets a B.A. or a B.S. after four years of study at one of the American universities I know intimately, is about as far advanced as the boy who, after two years of study, passes his intermediate examination at the E.T.H.

To see the full picture, we must realize that the minority of young people who pass the terminal examination (Reifeprürung, maturité, . . .) of the Swiss, or European, preparatory schools (Gymnasium, lycée, liceo, . . .) results from a *selection*. To begin with, only a minority of children is able to pass the entrance examination into the preparatory schools. Of those who, having surmounted this obstacle, enter the preparatory schools, a considerable percentage will be dropped along the road because of insufficient achievements. (About 50% in Zurich: of 100 children entering the preparatory school about 50 succeed in arriving at, and passing, the terminal examination.) Thus the preparatory school acts as a selecting agent, as a kind of sieve: it picks out the students who can be expected to deserve university education.

[It is obvious that any such procedure of selection, however beneficial it may be on the average, inavoidably involves hardships and undesirable results in some cases. It must be mentioned that such young people as do not succeed in the preparatory school have a second chance: they can attempt the "terminal examination" given by the state in regular intervals outside, and independently of, the preparatory schools (Eidgenössische Maturitätsprüfung, maturité fédérale). There are several private schools (some of them are rather good) preparing people for these state examinations.]

Some readers of this report may raise the question: How is the performance of the teachers in preparatory schools affected by the circumstance that only a very small percentage of their training has been devoted to didactical subjects? (*Cf.* Sec. 2.) This question was often in my mind when I spoke with teachers but I met just one teacher who was really interested in it and had a decided opinion. Here it is, in a nutshell: "As a student at the university, I was impressed

* This is, to some extent, actual practice as far as I can judge from the cases I know personally.

by some of my teachers. Then, when I started teaching in the school, I had difficulties. The trouble was, as I see it now, that I imitated my university professors: I did not realize that teaching in a school must differ in many ways from teaching at the university. The generalities I heard in our short course of general didactics gave me no help. I imagine that a good didactical course by a competent professor could have helped me to understand the situation sooner." Let us hope that my interlocutor was right.

4. Change. In this section, I speak only about the preparatory schools.

The program of the "state terminal examinations" (Eidgenössische Maturitätsprüfung; see Section 3) is fixed by a federal ordinance (of 1925). This program provides a minimum standard for the schools: the materials on this federal program are included in the curriculum of the schools, by law or by consent.

A committee of the Union of Swiss teachers of Mathematics and Physics recommended a carefully phrased new minimum standard for preparatory schools (in 1958; with the avowed purpose that the federal ordinance of 1925 should be correspondingly modified). The recommended changes are by no means revolutionary; here are a few.

For all three types, A, B and C: Less time should be spent on computations with logarithms, and the slide rule should be extensively used.

For the types A and B: Compound interest and especially annuities should be dropped, and the elements of the calculus introduced.

For the type C: Spherical Trigonometry should be dropped, and several other things introduced, especially vectors; geometric transformations should be treated, and so treated that they may serve as a preparation for the concepts of group theory.

These recommendations have been adopted by the schools to a good extent (most of them by the better schools, I am inclined to believe). At any rate, these recommendations have been widely discussed. There is a lively discussion and a strong interest in some schools, but the changes envisioned are by no means subversive. For instance, in the program of a school, mathematics and science gained a few semester-hours at the expense of Latin. Some teachers would like to teach a little more probability and add some statistics; others would insist more on geometric transformations and the group concept; and so on. In several schools, the programs are sufficiently elastic so that the teachers have a possibility to put their ideas into practice.

Some teachers in the French speaking Swiss cantons, especially in Geneva and Neuchâtel, would go much farther and, under French influence, they would like to "bourbakize" the curriculum; in fact, they have started to do so to some extent or will start next year. There were a few clashes; the "progressives" charge the others with "stabilism." I give the word to one of the "stabilists" who explained to me why he has objected so strongly to a talk by a "progressist": "He need not tell us what is a set, a ring, a field or a mapping, we have heard

all that and more at the university. He should have told us how and when, with which suggestive examples, to which use, he did, or intends to, introduce such concepts, and how much resistance he has encountered from the students."

I refrain, of course, from taking sides. I am inclined to believe that some "Bourbakism" will penetrate into the Swiss schools eventually, by degrees. On the other hand, it seems to me that the Swiss preparatory schools function, on the whole, satisfactorily, and so some "stabilism" is justified: experience and precaution are very much needed when new concepts or methods are introduced at the expense of tried and tested ones.

Appendix. Opinions such as have been voiced on the preceding pages cannot be "proved": they necessarily depend on the experience and the background of the author. Therefore, the reader is entitled to know through what experience the author has acquired his opinions.

I attended a school (in Budapest, Hungary) similar to the Swiss preparatory school of type A. I did practice teaching for a year in a school (also in Budapest) similar to the Swiss preparatory school of type C. In the Zurich preparatory schools of types A and B, I was part-time teacher for about three years around 1920. Later, for more than ten years, I was in charge of the mathematical part of the admission examinations to the E.T.H. (see Sec. 2) and there I examined candidates coming from schools all over Europe.

Since the fall of 1958 I spent about 8 months in Switzerland. I visited several schools, both in the German-speaking and the French-speaking parts, gave talks to pupils and teachers, and discussed their problems with the teachers. (Many of them attended my classes at the E.T.H. or I was well acquainted with them otherwise.) I spoke also with some key people in charge of the mathematical training of future primary and secondary teachers.

A MATHEMATICS INSERVICE EDUCATION PROJECT FOR ELEMENTARY SCHOOL TEACHERS

ROGER OSBORN AND M. VERE DEVAULT, The University of Texas

College mathematics departments will be assuming increasing responsibility for the success of elementary school mathematics programs as the rising demands for improvement spotlight the need, not only to improve the inservice teacher's methods, but also to augment his mathematical knowledge. This report of a project undertaken in Austin is being presented in—and should be justified by—the hope that such a report will (1) illustrate the possible cooperative role of the mathematics department of any college in improving elementary school mathematics, and (2) stimulate other mathematics departments to initiate like inservice courses in their communities to meet the rising tide of need and desire for such programs. The project was carried out jointly by the following University of Texas personnel: M. Vere DeVault, Associate Professor of Elementary Education; Roger Osborn, Assistant Professor of Mathematics;

W. Robert Houston and Claude C. Boyd, Research Associates. Its purpose was to increase the teacher's knowledge of mathematics in order that he might more successfully teach mathematics.

Procedure. In order most nearly to achieve the project's aim, it was thought mandatory to provide instruction both in mathematical content and in method, as the mathematics teacher's excellence was seen to be dependent upon a close union of mathematical knowledge and educational know-how.

The wide variance in the level of mathematical preparation of the teachers (from no college credit to fifteen semester hours), and in the length of time since they had received mathematical training, necessitated the use of elementary content for the programs. This content, however, was treated on a reasonably rigorous basis. The topics chosen for the five hour-and-a-half sessions of the course, and covered by means of closed-circuit TV, panel discussions, smaller group discussions, and lectures on mathematics and methods, were as follows:

(1) Numbers and numerals.
(2) Base and place (including bases other than 10).
(3) The structure of the number system and its relation to the four fundamental operations.
(4) Some laws of arithmetic (including the commutative, associative, and distributive laws and their uses in the elementary classroom).
(5) The use of models in teaching mathematics.

Effort was made to base the programs on experiences common to all teachers, as well as to give each session, insofar as was possible, a modern flavor. Also, attention was given to accommodating the participating teachers in that (1) no charge was made for the course, (2) the programs were presented on a night chosen by a majority of the teachers, (3) the sessions were begun and closed punctually, (4) no reports were required, and (5) previously prepared materials were given to each class member to minimize the amount of note-taking required.

Conclusions. Evaluation of each program and of the project as a whole was based on written comments from participating teachers representing the various Austin schools. Although not all were favorable, the majority of these informal appraisals indicated that the teachers judged the course to be of significant value and utility to themselves as teachers of mathematics. General conclusions reached as a result of the project were the following:

(1) Teachers recognize their need for improvement. (This fact was demonstrated by the enrollment alone, which had to be *limited* to 250.)
(2) Teachers will grasp at an opportunity for self-improvement.
(3) The level of effectiveness of the classroom teacher can be raised by even so limited a project as this.

LOGIC: GAME OR TOOL?

JON WHEATLEY, Victoria College, Victoria, British Columbia

A good many texts intended as an introduction to mathematics at a first-year university level now include what the authors are pleased to call "symbolic logic." It is this logic, its presentation and its function, that I wish to discuss here.

The usual treatment of symbolic logic in such texts is first to produce the truth tables for conjunction, disjunction and negation. The content of these tables is justified by a short exposition designed to show that this, in tabular, symbolic form, is the way we use "and," "or," "not" in the English language. The only reason authors get away with this (to me) obvious falsehood is that students at first-year level do not think critically about anything, least of all their own language. In the exposition it is usual to ignore the fact that the "inclusive" use of "or" in the language is relatively rare and that "and" usually carries a far heavier load of meaning than the truth table allows it; *e.g.*, consider the use of "and" in "He put out the cat and went to bed"—where it implies temporal succession. In my experience, students do not notice these discrepancies, but is logic introduced as an example of how large a camel they will swallow?

In some texts, the truth tables are followed by a list of axioms (usually incomplete). No mention is made of the relationship between the axioms and the truth tables except, sometimes, to set as problems the "proving" of the axioms! That is, not only are the fundamentals of the subject left in total obscurity but the function of an axiom set as basic to a system is implicitly denied. Finally, problems are set which assume that "and," "or," "not" as used in the English language are always used as logicians use their symbols and that material implication is translatable: "p implies q" or "if p then q."

These are certainly not all the muddles perpetrated by such treatments of symbolic logic, but they are a fair selection. They will suffice for my present purposes.

To me, one thing seems entirely certain in the teaching of elementary mathematics: that we should on no account give muddled accounts of anything to students, that mathematics, apart from its obvious utilitarian function, should parade as a triumph of clear thinking. I do not mean we should prove every theorem we ever use (impossible at a first-year level), nor that we should not foster good, practical mathematical intuition; I do mean that we should not present as obvious that which is false or that which is so inexplicit as to be rightly neither true nor false.

It should not be supposed from the foregoing that I am against teaching symbolic logic at an elementary level; I am, in fact, strongly for it. But I do not believe that the justification for doing so is that it introduces students to "modern mathematics." I feel that the justification is as follows: in the past, elementary mathematics courses were often solely manipulative in content. I

do not depreciate the value of this. However, I feel that two weeks to a month of a full course could profitably be devoted to (a) axiomatic methods and valid proof, (b) the relationship between mathematical system based on axioms and the world (of the physicist, of the chemist, of everyday life). It happens that symbolic logic is ideal for illustrative purposes in doing this and has the added advantage of providing an excellent entrance to set theory, which may be used in turn for counting and probability. The symbolism with which the student becomes familiar may also prove of value later.

For such a treatment of logic, I would suggest the following outline: First the elementary truth tables would be developed but it would be *emphasized* that the "and" and "or" of logic are heavily truncated when compared to ordinary language. Once the student has some facility with the truth tables, it would be pointed out that working with truth tables we are in a no-man's land; we have seen that the justification cannot be ordinary language and what else can it be? This offers an opportunity for a discussion of the function of an axiom set, followed by the presentation of an axiom set for symbolic logic with all the trappings, including meta- and object-languages and a light discussion of the difference between theorems within the system defined by the axioms and outside that system (meta-theorems). It would be insisted that although the psychological ground of the axiom set is the truth tables and though every axiom could be "proved" by the truth table method, the symbols used in the object language are uninterpreted. As the aim here would not be formal simplicity, a far longer and more complicated axiom set than necessary (on the model of that for the real numbers) would be best. On the basis of these axioms, various theorems could be proved with absolute validity. We then have a complete, elementary system of symbolic logic, securely founded on axioms and with the psychological origin of the axioms laid bare. Now, by interpreting this system in terms of propositions and connectives, the relationship between mathematics and the world can be discussed.

From this point the work can be extended. The idea of isomorphism (alternative interpretations of the object language) can be introduced with reference to switching circuits, Venn diagrams and set theory. If more complicated work of this type is desired, the set builder notation can be introduced and used to define quantified propositions. What happens at this stage is of far less interest to me here. My point is that with minor changes in what is taught, but major changes in how it is taught, some of the most important ideas of mathematics can be introduced without making the treatment unduly difficult. Then symbolic logic would become an important part of the student's training at an elementary level instead of being a rather poor game.

INSERVICE RE-EDUCATION OF MATHEMATICS TEACHERS

KENNETH E. BROWN, Office of Education

"Inservice re-education of mathematics teachers is a must if the improved mathematics programs are to be successful" was the conclusion of a recent conference on inservice education of mathematics teachers held at the U. S. Office of Education.

It was pointed out that the University of Illinois Committee on School Mathematics had been training teachers in the content and methods of teaching their program since its beginning in 1952. The School Mathematics Study Group, the University of Illinois Mathematics Project, the Ball State Teachers Project, the Commission on Mathematics of the College Entrance Examination Board, and others, recognize the need for re-education of teachers before the new and improved mathematics materials can be used effectively in the classrooms.

It is estimated that perhaps 90% of all the teachers need some re-education in mathematics. The conferees supported one speaker when he said "schools contemplating the introduction of new mathematics material should make adequate plans for a change by establishing an intensive inservice program. The inservice program should give the teachers the needed background for the new courses, and it should supply the mathematical and psychological reasons for the change. Without a carefully planned inservice education program, the introduction of new material into the classroom might end in disaster and subsequently prevent desirable changes."

The role of the administrator in inservice education programs was discussed. "Leadership is the key to the solution of our inservice problems" declared Dr. Henry Van Engen, specialist in teaching mathematics at the University of Wisconsin in an address to the conferees. "Superintendents and principals set the intellectual atmosphere in which a staff must work, and this atmosphere will condition the attitudes, the working habits, and the instructional goals of the staff. Although the supervisor should know the school and children, he should also know mathematics. This is particularly true of elementary supervisors. A sound mathematical program cannot be developed by one who is insecure about the mathematical principles to be taught."

A keynote speaker, Dr. W. L. Duren, Jr., Dean of the College of Arts and Science at the University of Virginia, stated that because of the failure to keep the high school mathematics content up-to-date "we find that we have been losing ground for thirty years. Despite their willingness to work, our school mathematics teachers do not have enough mathematical power to adapt to the demands of a new curriculum. If a student is not already pretty good in mathematics before he leaves high school, there is little that college mathematics can do to retrieve him as a scientist. College mathematics can add to the competence of those who have already achieved some mastery of school mathematics, but it cannot recover many who failed to get a good high school mathematical edu-

cation. Thus, the secondary school teacher is perhaps the key person in determining our national scientific and technological strength."

One discussion group concluded that "While there can be no one best plan to suit the needs of every local situation, any good inservice program may be described as one that:

1. Encompasses the needs of *all persons* engaged in the teaching of mathematics in the schools—teachers, supervisors and administrators.
2. Is organized on an all-school, or system-wide basis.
3. Is coordinated within school levels (elementary, junior high and high school) and from one grade to the next, through grade 12.
4. Is developmental in that it is continuous and adjustable to the needs of the classroom teacher and changes in curriculum development and implementation.
5. Recognizes the individual differences among teachers—differences that reflect preservice training, length and variety of teacher experience and grasp of subject matter.
6. Stimulates wholesome teacher growth and understanding of various viewpoints and competencies.
7. Makes provision for teachers to play some part in setting up the program of inservice training (surveys, suggestions and expression of preferences).
8. Is developed in an emotional climate which gives confidence to the participants and indicates administrative support.
9. Takes in account the needs and abilities of all the children—the slow learner, the retarded, the under-achiever, the average and the gifted.
10. Has the leadership of a competent and fair-minded administrator.
11. Is closely integrated with the supervisory and guidance program of the school.
12. Develops a variety of approaches to teacher growth (workshops, seminars, institutes, study groups, committee work).
13. Is judged for its effectiveness at regular intervals through various techniques of appraisal.
14. Draws upon the resources of the community, the colleges and universities and the professional staff itself.
15. Is aware of the limitations on teacher time and daily load and makes reasonable adjustments for teachers involved in the inservice program.
16. Gives recognition and encouragement to active participants in the inservice program in terms of promotional opportunities, financial remuneration, salary increases and professional status.

Other suggestions from the conferees included:

1. Released time from teaching for inservice courses such as the "Syracuse Plan" (New York).
2. Provision for inservice education film or TV kinescopes such as the Mathematics Kinescopes developed by New York City Board of Education.
3. Provision of additional consultants to guide and assist in inservice education programs.
4. Use of Skinner-type teaching machines for teaching improved mathematics, to be used by both student and teacher.
5. Use of visiting lecturers in mathematics such as the visiting lecturer program to secondary school teachers sponsored by the Mathematical Association of America.
6. Increase in inservice education for mathematics teachers through state organizations such as that now sponsored by the Illinois Council of Mathematics Teachers and California Council of Mathematics Teachers.
7. Also, television was suggested as a means of inservice education. The caution was added "it must be used wisely and with the proper teachers in charge."

Another discussion group suggested that the following questions be con-

sidered before an inservice education program be established:

1. Is leadership available either from a principal or superintendent?
2. Are consultants available; consultants for both mathematics content and organization and methods? This should include high school teachers as well as a college mathematician.
3. Is the climate in the school conducive to successful inservice education?
4. Does the teacher have a class using a modern program in mathematics? If not, is there one available for observation?
5. Does the teacher have time to study, organize, and experiment by:
    1. Teaching fewer classes?
    2. Being excused from some outside activities?
    3. Other means?
6. Is some financial support provided for:
    1. Textbooks and supplies?
    2. Consultant service?
    3. Extra teacher time or over-time pay?
    4. Making it possible for teachers to attend institutes, conferences and workshops?

Descriptions of many inservice education programs were given by the conferees. These materials will appear in the report of the conference. The conference, sponsored by the U. S. Office of Education and the National Council of Teachers of Mathematics, was attended by approximately fifty specialists who met in Washington, D. C., for two and one-half days.

NCTM Regional Conferences for School Administrators

Phillip S. Jones, president of the National Council of Teachers of Mathematics, has announced a Council-sponsored plan for sending consultants into the field to help school systems strengthen their mathematics programs through the use of the improved materials of instruction now available. The project, supported by a $48,000 grant from the National Science Foundation, is aimed at providing selected supervisors and school administrators with the information and orientation they need so that they may provide leadership in establishing improved mathematics programs in their school systems.

With this end in view, a series of eight regional invitational conferences has been scheduled during the fall of 1960. Each conference will be under the jurisdiction of a regional director and all conferences will be conducted by the same team of three consultants. This team consists of G. Baley Price, executive secretary of the Conference Board of the Mathematical Sciences; Kenneth E. Brown, mathematics specialist, U. S. Office of Education; and W. Eugene Ferguson, head of the Department of Mathematics, Newton High School, Newtonville, Massachusetts.

Each two-day conference has a four-part program: (1) "Progress in Mathematics and its Implication for the Secondary School," G. B. Price; (2) "The Drive to Improve School Mathematics—Comparisons and Common Elements of Special Programs," K. E. Brown; (3) "Our Experiences with the New Programs in Mathematics"—Panel of teachers to be selected in each region; (4) "Implementing the New Mathematics Program in Your School," W. E. Ferguson. In addition, time will be allowed for question sessions and for the inspection of the sample texts which will be on display. A preview of this program was presented at the Council's summer meeting at Salt Lake City, August 22, 1960.

The project is being directed by Frank B. Allen, member of the Council's Board of Directors and chairman of the Mathematics Department of Lyons Township High School and Junior College, La Grange, Illinois, with the advice of a steering committee including Max Beberman, E. G. Begle, Kenneth E. Brown, Edwin C. Douglas, Phillip

S. Jones, John R. Mayor, G. Baley Price, and Mina Rees. Ex officio members are Philip Peak and Henry Van Engen. Consultants to the committee are Harold P. Fawcett, Ohio State University, and Mary Hovet, supervisor of mathematics, Howard County Schools, Ellicott City, Md.

All Regional Directors have been appointed and the following schedule of Regional Orientation Conferences in Mathematics has been approved by the Steering Committee:

Date	*Location*	*Director*	*Region*
Oct. 3–4	Philadelphia, Pa.	M. Albert Linton, Jr., William Penn Charter School, Philadelphia, Pa.	Connecticut, Delaware, D. C., Maine, Maryland, Massachusetts, New Hampshire, New Jersey, New York, Pennsylvania, Rhode Island, Vermont.
Oct. 10–11	Iowa City, Iowa	H. Vernon Price, State University of Iowa	Illinois, Iowa, Minnesota, Nebraska, North Dakota, South Dakota, Wisconsin
Oct. 27–28	Atlanta, Georgia	H. Mack Huie, Atlanta Board of Education, Atlanta, Georgia	Alabama, Georgia, Louisiana, Mississippi, North Carolina, South Carolina, Virginia
Nov. 3–4	Portland, Oregon	William W. Matson, Mathematics Supervisor, Portland Public Schools, Portland, Oregon	Alaska, Idaho, Montana, Oregon, Washington, Wyoming
Nov. 18–19	Los Angeles, California	Clifford Bell, U.C.L.A.	Arizona, California, Hawaii, Nevada, Utah
Dec. 2–3	Topeka, Kansas	Marjorie L. French, Topeka High School, Topeka, Kansas	Arkansas, Colorado, Kansas, Missouri, New Mexico, Oklahoma, Texas
Dec. 9–10	Miami, Florida	Agnes Y. Rickey, Dade County Public Schools, Miami, Florida	Florida, Puerto Rico
Dec. 15–16	Cincinnati, Ohio	Mildred Keiffer, Supervisor of Mathematics, Cincinnati, Public Schools, Cincinnati, Ohio	Indiana, Kentucky, Michigan Ohio, Tennessee, West Virginia

The results of these conferences will be summarized in a brochure which is to be distributed by the National Council of Teachers of Mathematics.

Frank B. Allen

An Experiment in Individualized Mathematics in Grade 9

Queens College, the Division of Teacher Education, and Syosset High School in New York will start a research project in the teaching of mathematics in the 9th grade with the aid of a number of high schools in the New York metropolitan area. The experiments will be supported by a grant from the U. S. Office of Education. The object of the experiment is to develop methods and materials for individualized learning with a regular class so that pupils may advance at their own rate. The experiment will not be concerned with curriculum, but primarily with the use of perceptual materials. The development of the experiment is the result of work done by Dr. Gwladys C. Crosby of Queens College and Mr. Herbert Fremont, chairman of the mathematics department of Syosset High School, during the past two years.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1436. *Proposed by Mok-Kong Shen, Karlsruhe, W. Germany*

Through the vertices of a given $\triangle ABC$ draw straight lines l, m, n, respectively, such that n and l intersect in D, l and m in E, m and n in F inside $\triangle ABC$ and

(1) $\triangle ABE = \triangle BCF = \triangle CAD = \triangle DEF = (1/4)\triangle ABC$.

(2) $\triangle DEF$ is similar to $\triangle ABC$ and has an area equal to a given fraction of $\triangle ABC$.

E 1437. *Proposed by Nickolas Konopliv, University of Minnesota*

Test each of the following infinite series for convergence:

(1) $x+\sin x+\sin(\sin x)+\sin[\sin(\sin x)]+\cdots$,
(2) $x-\sin x+\sin(\sin x)-\sin[\sin(\sin x)]+\cdots$,
where $0<x<\pi$.

E 1438. *Proposed by S. H. Gould, Mathematical Reviews, Providence, R. I.*

A car travelling at a speed of v miles per hour is required by law to remain at a distance (in Rhode Island one car-length for every 10 miles per hour) of cv miles behind the car ahead, where c is a constant depending upon the state. Find, as a function of time, the maximum allowable acceleration for a car starting from rest immediately behind an unobstructed car which accelerates at a constant rate of a miles per hour per hour.

E 1439. *Proposed by J. M. Elkin, Long Island University*

Construct a bounded strictly monotonic function $f(x)$ such that $f'(x)$ exists for all real x and $\lim_{x\to\pm\infty} f'(x)\neq 0$.

E 1440. *Proposed by W. B. Carver, Cornell University*

Find a closed-form expression for $\sum_{s=1}^{n} s^3 r^{s-1}$, $r\neq 1$.

SOLUTIONS

Dissecting an Obtuse Triangle into Acute Triangles

E 1406 [1960, 290]. *Proposed by Michael Goldberg, Washington, D. C.*

Cut an obtuse triangle into the least number of acute triangles.

Solution by Wallace Manheimer, Franklin K. Lane High School, Brooklyn, N. Y. Let S be a set of acute triangles satisfying the problem. One line segment of S must divide the obtuse angle A, say, of the given triangle ABC. It may not terminate in the opposite side since otherwise another nonacute triangle would be formed and S would not be minimal. Therefore a vertex of S must lie within the triangle. At least five line segments of S must meet at this vertex, and a subset of S must form a centrally divided pentagon. It follows that S must contain at least seven triangles.

To construct a configuration with seven triangles, draw tangents DE and FG (D on AB, G on AC, E and F on BC) to the inscribed circle so that BDE and FGC will be acute isosceles triangles with vertex angles at B and C respectively. The bisectors of the angles of the pentagon $ADEFG$ will complete the required figure, since a triangle is acute if it contains two acute angles each greater than 45°.

Also solved by J. W. Baldwin, D. R. Bedgood, W. F. Cheney, Rufus Crane, Thomas Curry, R. E. Denman and V. E. Hoggatt (jointly), Jane Evans, P. J. Federico, C. E. Franti, L. D. Goldstone, D. E. Knuth, Harry Langman, D. C. B. Marsh, D. A. Moran, C. S. Ogilvy, S. W. Saunders, William Schoening, Walter Zayachkowski, and the proposer.

Federico gave an analysis of the problem (and also of that of dissecting a square into a minimum of eight acute triangles) based upon the Euler formula $V-E+F=1$ for a plane network. Denman and Hoggatt established the following allied results: (1) A nonacute triangle can be dissected into seven acute triangles; if $B>90°$, and $B-A<90°$ and $B-C<90°$, then the seven triangles can be isosceles. (2) An obtuse triangle can be dissected into eight acute isosceles triangles. (3) A right triangle can be dissected into nine acute isosceles triangles. (4) An acute nonisosceles triangle can be dissected into four congruent acute triangles or nine acute isosceles triangles. (5) An acute isosceles triangle may be dissected into four congruent triangles similar to the original triangle. (6) Any triangle can be dissected into two right triangles or three obtuse triangles.

The given problem was proposed by Martin Gardner in "Mathematical Games," *Scientific American*, vol. 202, Feb. 1960, p. 150, and a dissection into seven parts appeared in the Mar. 1960 issue, pp. 177–178.

An Application of Fermat's Little Theorem

E 1407 [1960, 290]. *Proposed by Leo Moser, University of Alberta*

Prove that, for $p>3$ and prime, ab^p-ba^p is divisible by $6p$.

I. *Solution by E. T. Parker and David Zeitlin, Remington Rand Univac, St. Paul, Minn.* Since $b(b^2-1)$ is a factor of b^p-b, we have $b^p-b\equiv 0 \pmod 6$. From Fermat's little theorem, $b^p-b\equiv 0 \pmod p$; and since $(6, p)=1$, it follows that $b^p-b\equiv 0 \pmod{6p}$. The addition of $ab^p-ab\equiv 0 \pmod{6p}$ and $-ba^p+ab \equiv 0 \pmod{6p}$ yields $ab^p-ba^p\equiv 0 \pmod{6p}$, the desired result.

II. *Solution by the proposer.* We will show more generally that $ab^p - ba^p \equiv 0 \pmod{\prod_{i=1}^{k} q_i}$, where the q_i are primes and $q_i - 1 \mid p-1$. The required result will then follow from the fact that $2-1 \mid p-1$, $3-1 \mid p-1$, and $p-1 \mid p-1$. To prove the more general result we need only show that q a prime and $q-1 \mid p-1$ implies $ab^p - ba^p \equiv 0 \pmod{q}$. If $ab \equiv 0 \pmod{q}$ the result is trivial, while if $ab \not\equiv 0 \pmod{q}$ then $a^{p-1} = a^{(q-1)t} \equiv 1 \pmod{q}$ and $b^{p-1} \equiv 1 \pmod{q}$ so that $(a^{p-1} - b^{p-1})ab \equiv 0 \pmod{q}$ in any case. Thus, for example, $ab^{61} - ba^{61} \equiv 0 \pmod{2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 31 \cdot 61}$.

Also solved by A. N. Aheart, Ray Authement, D. W. Bailey, J. W. Baldwin, Alan Beal, William Becker, Brother Joseph Heisler, Brother Louis Francis, T. C. Brown, James Burling, R. G. Buschman, Richard Cottle, D. F. Criley, Mary Ann Davis, Bob DeVore, J. W. Ellis, C. E. Franti, Anton Glaser, Michael Goldberg, L. D. Goldstone, R. Gramann, S. H. Greene, Bernard Greenspan, E. H. Grossman, Emil Grosswald, John Jordan, M. S. Klamkin, D. E. Knuth, Sidney Kravitz, J. P. Labute, Harry Langman, Joe Lipman, Gerald Leibowitz, Wallace Manheimer, D. C. B. Marsh, R. W. Means, M. V. Mielke, J. W. Moon, J. B. Muskat, Sidney Penner, D. J. Persico, J. L. Pietenpol, C. F. Pinzka, J. H. Rascoff, R. M. Rippey, Ray Rogers, R. T. Sandberg, William Schoening, J. A. Schumaker, D. L. Silverman, Vencil Skarda, E. L. Spitznagel, Jr., Robert Strichartz, E. A. Sturley, Carl Superko, R. P. Tapscott, L. Thomas, Dmitri Thoro, W. C. Waterhouse, Alan Wayne, Charles Wexler, Dale Woods, and W. Zayachkowski. Late solution by A. S. Gregory.

Mielke pointed out that the special case $p=5$ of this problem appears in *The Mathematics Teacher*, vol. LI, No. 8, Dec. 1958, p. 587, in the article "The Seventh Mathematical Olympiad for Secondary School Students in Poland." A labored solution for this case had been given by Olga Mitrinović in "Sur une congruence," *Bulletin de la Société des Mathématiciens et des Physiciens de la R. P. Macédoine*, t. vii, 1956.

The Highest Power of 2 in the Numerator of $\sum_{i=1}^{k} 1/(2i-1)$

E 1408 [1960, 290]. *Proposed by J. L. Selfridge, IBM, Yorktown Heights, N. Y.*

Find the highest power of 2 which divides the numerator of

$$1 + 1/3 + 1/5 + \cdots + 1/(2k-1).$$

Solution by D. L. Silverman, Defense Dept., Ft. Meade, Md. We have

$$1 + 1/3 + \cdots + 1/(2k-1) = \sum_{i=0}^{M-1} \{1/(i \cdot 2^{r+1} + 1) + 1/(i \cdot 2^{r+1} + 3) + \cdots + 1/((i+1) \cdot 2^{r+1} - 1)\},$$

where $k = 2^r M$, M odd. If in each of the M bracketed terms, the 2^r constituent terms are collected using as common denominator P_i, the product of the denominators within the ith term, then the resulting numerators will each consist of 2^r terms, each of the form P_i over a distinct odd residue of 2^{r+1}. These ratios must themselves be the odd residues of 2^{r+1} in some order. Since the sum of the odd residues of 2^N is 2^{2N-2}, it follows that each of the M numerators is of the form $2^{2r} M_i$, M_i odd. The numerator of $1+1/3+\cdots+1/(2k-1)$ is thus of the form $2^{2r}Q$ where Q is the sum of an odd number of odd terms, hence odd. Hence

if 2^r is the highest power of 2 dividing k, 2^{2r} is the highest power of 2 dividing the numerator of $1+1/3+\cdots+1/(2k-1)$.

Also solved by J. P. Ballantine, Leonard Carlitz, N. J. Fine, D. E. Knuth, D. C. B. Marsh, R. T. Sandberg, and the proposer.

Ballantine proved, more generally, that if $S(n, k)=\sum_{i=1}^{k}(2i-1)^n$ and if, for $N\neq 0$, $T(N)$ is the exponent of the highest power of 2 that divides N, and if $T(N/D)$ is defined as $T(N)-T(D)$, then $T(S(n, k))=T(k)$, if $T(n)>0$; $T(S(n, k))=2T(k)$, if $T(n)=0$. The answer to the given problem is found by taking $n=-1$.

The proposer pointed out that the given problem is left unsolved in H. S. Shapiro and D. L. Slotnick, "On the mathematical theory of error-correcting codes," *IBM J. Res. Develop.*, v. 3 (1959), p. 31 (Remark 1). There the authors say, "an estimate [of this] power of 2 $\cdots$ seems in general to be a difficult number theoretic problem."

Bidding Sequences in Bridge

E 1409 [1960, 290]. *Proposed by Paul Lukacs, Tel-Aviv, Israel*

In the game of bridge North is dealer. (a) What is the number of the different bidding sequences, assuming that West and East are silent? (b) What is the number of the different bidding sequences if all four players must participate in the bidding?

Solution by Walter Zayachkowski, University of Alberta. (a) North has 36 possible opening bids. The number of different bidding sequences is $2^{36}-1$, the number of subsets of $\{1, 2, \cdots, 36\}$ excluding the null subset.

(b) Labelling 7 No Trump as bid 1, 7 Spades as bid 2, $\cdots$, 1 Diamond as bid 34, 1 Club as bid 35, all possible opening bids, if North opens with bid k we have $7\cdot 22^{k-1}$ different bidding sequences. The total number of different bidding sequences for the 35 opening bids is $(22^{35}-1)/3$. Should North pass originally, there are an additional 22^{35} bidding sequences, giving a total of $(4\cdot 22^{35}-1)/3$ different bidding sequences.

Also solved by L. D. Goldstone, W. C. Guenter, A. R. Hyde, D. E. Knuth, Wallace Manheimer, D. C. B. Marsh, J. L. Pietenpol, D. L. Silverman, and the proposer.

Not all the above solutions agreed. For an allied problem see E 801 [1948, 578].

Isomorphic Groups

E 1410 [1960, 290]. *Proposed by Albert Wilansky, Lehigh University*

Is there a field whose additive and multiplicative groups are isomorphic?

I. *Solution by W. J. Blundon, Memorial University of Newfoundland.* No. Because there is implied the existence of a field in which the usual postulates hold and also in which addition is distributive over multiplication. Let 0 be the additive identity, e the multiplicative identity, and x any element of the field. Then $e+0\cdot x=(e+0)(e+x)$ gives $x=0$, which contradicts the fact that a field must contain at least two distinct elements.

II. *Solution by N. J. Fine, Institute for Advanced Study.* No. The equation

$x^2=1$ has either one or two solutions, depending on whether $-1=1$ or $-1\neq 1$, that is, on whether $2=0$ or $2\neq 0$. In the first case the corresponding equation $2x=0$ has at least the two solutions $x=0$, 1, and in the second case it has only one solution, $x=0$.

Also solved by Neil Bernstein, D. M. Bloom, J. L. Brown, Jr., T. C. Brown, J. N. Deeter, R. H. DeVore, J. W. Ellis, R. L. Faber and M. I. Freedman (jointly), E. R. Gentile, L. D. Goldstone, J. D. Haggard, Shigeru Ishii and Joel Levy (jointly), John Jordan, M. S. Klamkin, D. E. Knuth, A. G. Konheim, J. P. Labute, Loren Larson, Joe Lipman, Gerald Losey, J. A. Lutts, D. C. B. Marsh, H. F. Mattson, D. R. Morrison, F. D. Parker, D. J. Persico, J. L. Pietenpol, R. E. Priest, Azriel Rosenfeld, C. P. Sequin, D. L. Silverman, C. F. Stephens, Wu Ta-Sun, Seth Warner, and Alan Wayne.

This problem is the same as Problem 4644 [1956, 587]. In the last line of the solution given there, the words "one to one" should be replaced by the word "isomorphic."

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers, The State University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers, The State University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4929. *Proposed by James W. Brown, University of Michigan*

Given that $y(t)$ satisfies the integral equation

$$y(t) = t^k + \int_0^t y(t-\tau)J_0(\tau)d\tau,$$

where $J_0(t)$ is the first order Bessel function and $k>-1$. Show that

$$\int_0^\infty e^{-t}y(t)dt = \frac{(\sqrt{2})\Gamma(k+1)}{\sqrt{2}-1}.$$

4930. *Proposed by D. F. Rearick, University of British Columbia*

For a fixed positive integer n, consider the numbers $a_m=(m, n)$ as m ranges over a complete residue system (mod n). Prove that the number of a_m having an even number (including none) of distinct prime factors exceeds the number having an odd number of distinct prime factors if n is odd, and that the two are equal if n is even.

4931. *Proposed by Harry Goheen, Oregon State College*

In an algebra for which there is a commutative law but no associative law, how many formally distinct sums are there of n elements?

4932. *Proposed by D. S. Mitrinovitch, Belgrade University, Yugoslavia*

Consider the differential equation $z^2(d^4w/dz^4)=a(d^2w/dz^2)$, where a is a real parameter, $z=x+iy$. Determine the singularities of the complex functions $w=f(z, a)$ which satisfy it.

4933. *Proposed by I. N. Herstein, Cornell University*

Suppose a ring R is the set theoretic union of a finite number of commutative fields; prove that R must then be a commutative field.

4934. *Proposed by I. N. Herstein, Cornell University*

Let F be a field of characteristic 0, let G be a group, and $\Gamma(G, F)$ the group algebra of G over F. Prove that $\Gamma(G, F)$ is an algebraic algebra over F if and only if G is locally finite.

SOLUTIONS

Definite Integrals

4877 [1959, 921]. *Proposed by O. P. Aggarwal and Irwin Guttman, University of Alberta*

Show that

$$\int_0^a e^{-t^2/2}dt = (\pi/2 - aI)^{1/2},$$

where

$$I = \int_{a^2}^{2a^2} \frac{e^{-\omega/2}d\omega}{\omega\sqrt{(\omega - a^2)}}.$$

I. *Solution by E. S. Keeping, University of Alberta.* Let $N=\int_0^a e^{-t^2/2}dt$, $a>0$. Then

$$N^2 = \int_0^a \int_0^a e^{-(t^2+u^2)/2}dtdu.$$

Changing to polar coordinates and noting the symmetry about $\theta=\frac{1}{4}\pi$, we obtain

$$N^2 = 2\int_0^{\frac{1}{4}\pi} \int_0^{a\sec\theta} e^{-r^2/2}rdrd\theta = 2\int_0^{\frac{1}{4}\pi} (1 - e^{-(a^2\sec^2\theta)/2})d\theta.$$

Putting $\omega=a^2\sec^2\theta$, $d\omega=2a^2\sec^2\theta\tan\theta d\theta=(2\omega/a)(\omega-a^2)^{\frac{1}{2}}d\theta$, we get the desired result,

$$N^2 = \tfrac{1}{2}\pi - a\int_{a^2}^{2a^2} \frac{e^{-\omega/2}d\omega}{\omega\sqrt{(\omega - a^2)}}.$$

II. *Solution by Marlow Sholander, Carnegie Institute of Technology.* Let

$$P = \int_0^a e^{-t^2/2}dt, \qquad f(w) = \frac{e^{-w/2}}{w\sqrt{(w-a^2)}}, \qquad I = \int_{a^2}^{2a^2} f(w)dw.$$

We integrate the last by parts, differentiate as shown, and substitute $w=t^2+a^2$, obtaining:

$$aI = \tfrac{1}{2}\pi e^{-a^2} + \int_{a^2}^{2a^2} e^{-w/2}\cos^{-1}\frac{a}{\sqrt{w}}\,dw,$$

$$\frac{d(aI)}{da} = -\int_{a^2}^{2a^2} wf(w)dw = -2e^{-a^2/2}P = -\frac{dP^2}{da}.$$

Hence $aI+P^2=\frac{1}{2}\pi$, the value at $a=0$.

Also solved by A. R. Bradley, D. A. Breault, Robert Breusch, J. W. Brown, R. G. Buschman, L. Carlitz, P. R. Chernoff, P. G. Engstrom, N. J. Fine, Todd Gitlin, M. L. Glasser, George Glauberman, Emil Grosswald, M. S. Klamkin, A. G. Konheim, T. V. Lakshminarasimhan, S. Leja, Julius Lieblein, D. C. B. Marsh, J. S. Potter, Barbara Sakitt, E. M. Scheuer, Chih-yi Wang, Robert Weinstock, David Zeitlin, and the proposers. Late solution by E. K. McLachlan.

Helly's Theorem

4878 [1959, 921]. *Proposed by D. J. Newman, Brown University*

A collection of closed bounded convex sets is given in the plane with the property that any three of them have a point in common. Prove they all have a point in common.

Solution by D. S. Greenstein, University of Michigan. The stated proposition is case $n=2$ of Helly's theorem, an elegant proof of which is to be found in H. Rademacher and I. J. Schoenberg, *On Helly's theorem on convex sets and the Tchebycheff approximation problem*, Canad. J. Math., 1950. Helly's original proof appeared in Jber. Deutsch Math. Verein., vol. 32, 1923, pp. 175–176.

Also solved by S. H. Eisman, Irma Esrig, Emil Grosswald, D. W. Henderson, Nicholas Kazarinoff, P. S. Landweber, Yoshio Matsuoko, Marlow Sholander, Robert Weinstock, A. S. Zamanakos, and the proposer. Late solution by E. K. McLachlan.

A Definite Integral Relationship

4880 [1959, 921]. *Proposed by A. W. Goodman, University of Kentucky*

Show that, for all α, β, $0<\alpha<\beta<\pi$,

$$\int_0^\alpha \sqrt{\left(\frac{\cos\theta-\cos\beta}{\cos\theta-\cos\alpha}\right)}\,d\theta + \int_\beta^\pi \sqrt{\left(\frac{\cos\beta-\cos\theta}{\cos\alpha-\cos\theta}\right)}\,d\theta = \pi.$$

Solution by Leonard Carlitz, Duke University. If we put $x=\cos\theta$, $a=\cos\alpha$, $b=\cos\beta$, $-1<b<a<1$, the stated result becomes

$$(1)\qquad \int_{-1}^{b} \sqrt{\left(\frac{b-x}{a-x}\right)} \frac{dx}{\sqrt{(1-x^2)}} + \int_{a}^{1} \sqrt{\left(\frac{x-b}{x-a}\right)} \frac{dx}{\sqrt{(1-x^2)}} = \pi.$$

Since, when $b=a$ (1) obviously holds, it will suffice to show that the derivative (with respect to b) of the left member vanishes, *i.e.*:

$$(2)\qquad \int_{-1}^{b} \frac{dx}{\sqrt{\{(a-x)(b-x)(1-x^2)\}}} - \int_{a}^{1} \frac{dx}{\sqrt{\{(x-a)(x-b)(1-x^2)\}}} = 0.$$

We apply the transformation $y=(\lambda x+1)/(x+\lambda)$, choosing λ so that $(a+b)\lambda^2 + 2(ab+1)\lambda+(a+b)=0$, $|\lambda|>1$. Since the discriminant of the quadratic is $(1-a^2)(1-b^2)>0$, λ is real. Put

$$k = \frac{\lambda a + 1}{a+\lambda} = -\frac{\lambda b + 1}{b+\lambda};$$

then $0<k<1$ and the intervals $[-1, b]$, $[a, 1]$ are carried into $[-1, -k]$, $[k, 1]$, respectively. The first integral in (2) now becomes

$$\left(\frac{\lambda^2-1}{(\lambda+a)(\lambda+b)}\right)^{\frac{1}{2}} \int_{-1}^{-k} \frac{dy}{\sqrt{\{(y^2-k^2)(1-y^2)\}}}.$$

The transform of the second integral in (2) is the same except that the limits of integration are k to 1. Thus they are equal, and their difference vanishes, as required.

Also solved by Marlow Sholander using contour integration.

Entire Function with Special Property

4881 [1960, 86]. *Proposed by Burton Randol, The Rice Institute*

Is there a nontrivial entire function f, such that $\int_L f(z)dz$ vanishes for all straight lines L infinite in both directions?

Solution by J. C. Kneser, Tübingen, Germany. A nonconstant entire function which tends to zero along every half-line has been exhibited by G. Mittag-Leffler, Acta Math. vol. 42, 1920, pp. 285–308. The derivative of this function has the characteristics required by the present problem.

More general examples may be found in K. Grandjot, Math. Ann. vol. 91, 1924, pp. 316–320 and H. Bohr, S.-B. Preuss. Akad. Wiss., Phys.-Math. Kl. 1929, 565–571.

Also solved by W. H. J. Fuchs and the proposer.

Filters with the Countable Intersection Property

4882 [1960, 86]. *Proposed by I. S. Gal, Cornell University*

An (ω, ∞)-filter in a set X is a filter $\mathfrak{F}$ in X which has the countable intersection property: $F_n \in \mathfrak{F}$ implies $\bigcap F_n \in \mathfrak{F}$ $(n=1, 2, \cdots)$. Prove the following proposition:

If X is a metric space and if every (ω, ∞)-filter in X has a nonvoid adherence then the same holds for every open subspace of X.

Solution by Meyer Jerison, Purdue University. The following is the essential content of problem 8H.5 in L. Gillman and M. Jerison, *Rings of Continuous Functions.*

THEOREM. *For any topological space X, every filter on X with the countable intersection property has nonvoid adherence if and only if X is a Lindelöf space, i.e., every open cover of X has a countable subcover.*

The proof is like the proof of the theorem that every filter on a space has nonvoid adherence if and only if the space is compact—"finite" being replaced by "countable."

Now, a metric space is a Lindelöf space if and only if it is separable. In fact, for each positive integer n, a metric Lindelöf space is covered by countably many balls of radius $1/n$, and the centers of all such balls constitute a countable dense set in the space. The converse follows directly from the existence of a countable base of open sets in a separable metric space.

Thus, the class of spaces considered in the problem is the class of separable metric spaces, and every subspace of such a space, whether open or not, belongs to the class.

Also solved by the proposer.

Symmetric Continued Fraction

4883 [1960, 86]. *Proposed by W. A. Schneider, Milwaukee, Wisconsin*

If

$$F = 1 + \frac{1}{a_1+}\,\frac{1}{a_2+}\cdots\frac{1}{a_n+}\,\frac{1}{a_n+}\cdots\frac{1}{a_2+}\,\frac{1}{a_1}$$

and P_n/Q_n is the nth convergent of F, prove

$$\cot^{-1} Q_{2n} = \cot^{-1} P_{2n} + \cot^{-1} P_{2n+1}.$$

Solution by C. F. Pinzka, University of Cincinnati. From the recurrence relations, $Q_{2n+1}=a_1Q_{2n}+Q_{2n-1}$, $Q_{2n}=a_2Q_{2n-1}+Q_{2n-2}$, $\cdots$, $Q_3=a_2Q_2+Q_1$, $Q_2=a_1Q_1$, we have

$$\frac{Q_{2n+1}}{Q_{2n}} = a_1 + \frac{1}{a_2+}\cdots\frac{1}{a_n+}\,\frac{1}{a_n+}\cdots\frac{1}{a_2+}\,\frac{1}{a_1}$$

$$= \frac{1}{F-1} = \frac{Q_{2n+1}}{P_{2n+1}-Q_{2n+1}}.$$

This gives $Q_{2n}=P_{2n+1}-Q_{2n+1}$. Eliminating Q_{2n+1} from this last relation and $P_{2n+1}Q_{2n}-P_{2n}Q_{2n+1}=-1$, we have

$$Q_{2n} = \frac{P_{2n}P_{2n+1} - 1}{P_{2n} + P_{2n+1}},$$

which is equivalent to the proposed result.

Also solved by Evelyn Frank, Robert Heller, Stanton Philipp, J. H. Toope, and Chih-yi Wang. Late Solution by S. Parameswaran.

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association

Mathematical Analysis. By Edwin M. Hemmerling. McGraw-Hill, New York, 1959. xi+332 pp. $5.75.

This book presupposes that the student has completed his full three years of normal high school mathematics and that he will follow this book with a study of analytic geometry and calculus. The conceived function was to review high school mathematics. Thus the author does not feel obligated to develop the sections in a logical sequence (trigonometry is used long before the definition is given), or include topics such as word problems, binomial theorem or inequalities.

In those cases where a student lacks the proficiency to master the calculus course, it is wasteful of time, talent and tuition to compound the difficulty further by repetition of poorly organized remedial drill on algebraic techniques that are deficient in clarity, significance or logical justification. This book's method of presentation is limited to a couple of worked examples per section and a paucity of proofs. Although the author states "Particular emphasis is given to the analytic relationship between trigonometric functions," the student is referred elsewhere for proofs of the addition formulas.

In a fruitless attempt to justify the title, the last two chapters are concerned with infinite series and differential calculus. The common mistakes about limits and the chain rule are present. It is no wonder that high schools are advised not to attempt a discussion of calculus with such inaccurate texts being written for college use.

From among other misprints, mistakes of carelessness and misleading statements, the following are taken as samples: no assumption regarding the rationality of the coefficients precedes the conclusion that the roots are rational if b^2-4ac is a perfect square; the number of inversions in 2431657 is miscounted; and $r=.013$ is used in converting $.61313\cdots$ to rational form.

JAMES H. MCKAY
Michigan State University Oakland

Fundamental Principles of Mathematics. By John T. Moore. Rinehart, New York, 1960. 630 pp. $7.00.

Here is an interesting agglomeration of "standard" and "modern" topics intended for a two semester college freshman course, either a pre-calculus course or a terminal course.

To give an air of flavor there follows a sketchy sampling of the vocabulary —roughly in the order of first occurrence:

> Proper subset, disjoint, power set, cardinal number, cartesian set, intersection, complement, Boolean algebra, reflexive, equivalence relation, independent postulates, natural numbers, numeral, commutative, monotonic law, rational numbers, real numbers, sequence set, nested intervals (we are on page 39 now!), exponent, principal root, scientific notation, significant digit, logarithm, relation, ordered pair, domain, function, cartesian coordinate, inverse relation, power function, hyperbola, asymptotes, slope of a line, basic circular functions, periodic function, Cap-sine (inverse is function), quadratic equation, completing the square, point-slope equation for line, one parameter family, distance between two points, trigonometric identities, degree circular functions, radian (we are ½ through; remaining terms are general and more sparsely chosen), trigonometry of the general triangle; glimpse of differential calculus; polynomial function, maximum-minimum; glimpse of integral calculus; complex numbers, fundamental theorem of algebra; real polynomial equations; inequalities; systems of equations, matrices, determinants; plane analytic geometry; polar coordinates; solid analytic geometry; binomial theorem; probability; statistics.

Obviously not all of the topics alluded to above can be treated thoroughly. The specialist in foundations or other purist who believes that a freshman should be told "the truth" will often be horrified. Much of the exposition will be understandable only to students who have already learned the material either in previous courses or in class. Nevertheless, if a book along these lines is indicated, this is a generally creditable job. Probably its strongest feature is the large number of excellent exercises.

DAVID A. PAGE
University of Illinois

Real Variables. By John M. H. Olmsted. Appleton-Century-Crofts, New York, 1959. xvi+621 pp. $9.00.

This work is an extension of Olmsted's *Intermediate Analysis*, which constitutes the first nine chapters of this book. It contains topics normally covered in a first course in advanced calculus. The author has starred certain sections and exercises and double starred others. The starring indicates the level of the work, the unstarred portions being the least difficult, the starred ones more so, and so on. For example, the work which deals with uniform continuity on convergence is in starred sections while sections involving topics such as metric topological spaces are doubly starred. Line integrals are introduced solely in doubly starred exercises while work on the Legendre and Tchebycheff polynomials are introduced in doubly starred sections. Thus this text could be used for a first course in analysis at various degrees of difficulty.

There is an abundance of exercises varying in difficulty from the graphs of simple functions to proofs of the Minkowski inequality for finite forms. The proofs and extensions of many theorems are relegated to the exercises. Thus, exercises are available for the average student and enough of greater difficulty to stimulate the above-average student. Throughout the text the author frequently refers to the exercises and it is sometimes disconcerting when dealing with an unstarred topic to find a reference to one that is starred. As a result, a teacher must take care in assigning exercises to see that those are assigned which are essential to the development of the theory he desires to present.

The book is well written with numerous examples and should prove satisfactory as a text in courses called advanced calculus or introduction to the theory of functions of a real variable.

R. G. SANGER
Kansas State University

The Analysis of Variance. By H. Scheffé. Wiley, New York, 1959. xvi+477 pp. $14.00.

This book presents the theory of the analysis of variance and is not a cookbook, though computational methods are outlined and sufficient problems included to give the reader ample opportunity for practice. The author stresses vector space and geometrical ideas; for instance, the sum-of-squares decomposition may be viewed as resolving the observation vector into components lying in specified subspaces. In the opinion of the reviewer, this book is more comprehensive and complete than its predecessors, and is recommended to all serious students of statistics. It is suitable for either a course at the advanced undergraduate or graduate level, or for self-study. The author suggests as necessary only a minimum mathematics background, but the reviewer strongly recommends a course, or its equivalent, in linear algebra for potential readers of this very fine book.

The contents include the theory of estimation and least squares, and the Gauss-Markov theorem in Chapter 1; the ideas of confidence ellipsoids, tests, and the likelihood ratio are in Chapter 2. Subsequent chapters discuss the analysis of designs ranging from the simple one way to incomplete blocks. Chapter 6 is about covariance. The next three chapters present the analysis of random, mixed, and randomization models. In Chapter 10 he discusses the effect of certain departures from the underlying assumptions. Six appendices deal with vector algebra, matrix algebra, ellipsoids, noncentral distributions, the multivariate normal, and Cochran's theorem. Tables of F, its power, and the studentized range are included. There are numerous literature citations, a long list of references, and a profusion of interesting footnotes.

THOMAS E. KURTZ
Dartmouth College

Introduction to Mathematical Physics. By William Band. Van Nostrand, Princeton, N. J., 1959. x+326 pp. $7.25.

This stimulating work is a well-organized collection of brief surveys of topics in mathematical physics, all of which are currently of cardinal interest and importance. It was designed primarily to supply a preparation for senior and graduate students for more penetrating study in the advanced special treatises. In the reviewer's opinion it does fulfill this purpose very well. It is not a difficult book to read and the author has been able to skim off a considerable amount of the cream from an impressive variety of subjects. The style is relaxed and easy-going and relief from the technical details is provided occasionally by historical and philosophical material. Considerable emphasis is placed on the heuristic value of invariance but full use is not made of this powerful concept. The rigor level of the mathematical developments is about the average for books of this type.

Most works on mathematical physics are not uniformly modern so far as the real variable aspects are concerned. The limit of an infinite numerical process, for example, may be identified with a stage of the process rather than taken as a number associated with the process in a certain way. There are one or two passages in the present text which admit of valid interpretations but nevertheless might tend to strengthen the misconceptions of students who have not had rigorous courses in the calculus, for example, (page 85) . . . "the partial derivative $\partial v/\partial t$ refers to a fixed position in space, so that we are comparing the velocities of two different pieces of material at the two close instants of time." Also in one or two places the student might be led to the erroneous conclusion that the Euler equations constitute sufficient conditions for an extreme value, and the statement (page 4) "The relations between length and area necessitate the use of irrational numbers $2^{\frac{1}{2}}$, pi (π), etc." although true needs revision. These are, of course, not serious matters for the student who is well prepared in mathematics.

This book naturally consists of two parts: the mathematical background and the mathematicophysical theories. Because of limitations of space the mathematical developments are necessarily sketchy. As a time-saving device, vector algebra is presented at the outset by means of a framework of definitions that have their origins in the ordinary geometrical vector algebra, specifically, "The vector is here defined as the set of three numbers, considered as a whole and obeying a certain algebra which consists of mutually consistent rules for adding, multiplying, differentiating such entities." . . . "The *scalar product* of two vectors written $\mathbf{A}\cdot\mathbf{B}$, is defined by $\mathbf{A}\cdot\mathbf{B}=A_1B_1+A_2B_2+A_3B_3$" . . . "The angle θ between two vectors $\mathbf{A}$ and $\mathbf{B}$ is defined by $\cos\theta=(\mathbf{A}\cdot\mathbf{B})/AB$" . . . . A set of three ordered real numbers (*i.e.*, a function defined over the discrete set 1, 2, 3) is not the same mathematical concept as a vector in the sense of a directed line segment characterized by postulates extracted from euclidean geometry. A vector in the latter sense has associated with it an infinite collec-

tion of sets of components (one or two sets for each coordinate system). Because of the conceptual differences verifications must be made (if understanding is to be above the manipulative level) before the formulas of the strictly numerical theory can be employed in geometric situations. These verifications are left to the student but pertinent detailed figures and problems are supplied.

Among the topics treated in the part of the book concerned with physical theories are: vibrating strings and membranes, elasticity theory, fluid dynamics, Lagrangian and Hamiltonian mechanics, canonical transformations, phase space, statistical mechanics, electromagnetic theory, Lorentz transformations, invariance of electromagnetic laws, gravitation in general relativity, wave-particle duality, Schrödinger's equation, and relativistic quantum mechanics. There is a slip of the pen in the part on relativity. The author finds it convenient because of previous notational commitments to denote contravariant quantities by subscripts and covariant quantities by superscripts—a reversal of the usual notation. However, on page 283, the expression for interval is written $g_{\mu\nu}dx_\mu dx_\nu$ and here the $g_{\mu\nu}$ cannot be contravariant because the form is invariant. However, it should be borne in mind that the presence of anomalies in advanced original works is almost inevitable and the present text does have many marks of expositional originality. In the reviewer's opinion this book is a valuable contribution to the literature.

HOMER V. CRAIG
Boeing Airplane Co. and The University of Texas

Methods Based on the Wiener-Hopf Technique. By B. Noble, Pergamon Press, New York, 1959. x+246 pp. $10.00.

This book is designed for applied mathematicians, and contains an excellent selection of solutions to mixed boundary value problems from such fields as hydrodynamics, acoustics, electromagnetic theory and potential theory. Throughout the text the differential equations themselves are transformed, thus avoiding the use of too many special functions, Green's functions, and integral equations. This simplified approach (attributed to D. S. Jones) uses only the original differential equations, and the Fourier, Laplace or Mellin transformation applied to these equations, with an application of analytic continuation of the transforms.

There is included a section on approximate solutions, and a section pointing up the connection of the Wiener-Hopf method with the general Hilbert-Riemann problem.

On the whole, there is enough attention to rigor to indicate the appropriate steps, but not so much that the text is bogged down with too many fine details. This makes the book very readable. Further, the discussion of a large number of research papers makes the book an excellent reference for workers in this field.

GORDON LATTA
Stanford University

Finite Mathematical Structures. By John G. Kemeny, Hazleton Mirkil, J. Laurie Snell, and Gerald L. Thompson. Prentice-Hall, Englewood Cliffs, N. J., 1959. xi+487 pp. $7.95.

This book has many points of resemblance to the earlier *Introduction to Finite Mathematics*, by three of the present authors. But where that book was intended for biological and social science students, this one is aimed at the physical sciences and engineering. Furthermore, the earlier book was for freshmen, while this one is for sophomores. There are seven chapters: 1. Compound statements (propositional calculus, truth tables, etc.), 2. Sets and functions, 3. Probability theory, 4. Elementary linear algebra, 5. Convex sets (including in particular an introduction to linear programming), 6. Finite Markov chains, 7. Continuous probability theory. Practically everything starts off in the most elementary way possible, and a knowledge of the calculus is assumed at only a few points, these mostly in the last chapter. The book is designed for a year course, or for a one-semester course in either probability or linear algebra. The book would be less satisfactory as an algebra text than as a probability text, since it doesn't get far in linear algebra. Despite the fact that Chapter 4 is the longest, the flavor of the book is not overly algebraic; elementary logic, probability and related topics get a more thorough treatment.

It is of course refreshing to see a book like this, which recognizes the necessities of the modern world: probability, linear algebra, and linear programming, for instance, are at least as valuable to the physical scientist and engineer today as the subjects we usually stuff them with, such as tricky differential equations. And, by and large, the authors have done a good job of introducing these subjects. The reviewer is impressed with the way they have managed to bring in important mathematical concepts and problems; the continuum hypothesis and Goldbach conjecture, for instance. There are a profusion of problems, mostly attractive and informative, and quite a few are novel.

The reviewer's reservations are of several kinds. First, an even casual inspection reveals statements like: (p. 54) "It would be difficult to give a simple description for the subset containing the elements $\{P1, P4, P14, P30, P34\}$." (In context, this looks better than it does here, but perhaps that makes the sentence even more insidious.) Or, discussing how a function can be defined (p. 71), "And a formula is possible only if both D and the range R consist of numbers." (In Chapter 4, for instance, they give what *look* like formulas for functions on vector spaces.) Second, a good argument can be made that mathematics majors, at least, should not be exposed to truth tables, Venn diagrams, propositional calculus, etc., until they know more honest-to-God mathematics. The treatment must of necessity be elementary and simple (as it is here), yet there is something facile and flashy about such matters which can mislead the student. (By the way, the book does not lend itself easily to the omission of the first two chapters, since the terminology introduced there is used heavily in later chapters. This has the interesting side effect that some statements about

very basic mathematics, later on, sound mysterious and strange to adult mathematicians.) The authors may have discovered that students are not misled, in fact, by their treatment; it is a pragmatic fact about which the reviewer has no definite knowledge.

The reviewer's final feeling: it would be pleasant to try this book on sophomores, but playing down part of the first two chapters, and introducing a more abstract and thorough treatment of linear algebra towards the end of Chapter 4. If one were willing to forego some of the later material on probability, the book might even be suitable for a class of better-than-average freshmen, who presumably know no calculus. Almost certainly this text is a glimpse into the undergraduate near future, with less emphasis than today on gobs of calculus, differential equations, and related analysis topics.

D. R. HUGHES
University of Chicago

Statistical Independence in Probability, Analysis and Number Theory. By Mark Kac. Carus Monograph No. 12. Mathematical Association of America, 1959.* Distributed by Wiley, New York. xiv+93 pp. $4.00.

This book is an outgrowth of the 1955 Hedrick Lectures in which, the author states, his aim was to show that "(a) extremely simple observations are often the starting point of rich and fruitful theories and (b) many seemingly unrelated developments are in reality variations on the same single theme." This is well done indeed in this beautifully written little book.

Starting with the idea of independence implied in a formula of Vieta, which in terms of Rademacher functions shows a case where the integral of a product equals the product of integrals, Kac proceeds through a discussion of the vague early notions of probability and independence, Borel's work marking the beginnings of modern probability theory, and the later developments of this work. He puts such ideas as set-theoretical vs. frequency definitions of probability into perspective, discusses several "laws of large numbers" in terms of Rademacher functions and Lebesgue measure and indicates their connection with "the normal law." Comments and examples tying these ideas to other realms of mathematics and physics abound.

A chapter on the density of prime numbers and the normal law in number theory is followed by one relating the ergodic theorem to continued fractions.

This book should be required reading for students in probability courses and is highly recommended to others as instructive in perspective, methods of proof, lessons to be learned from history and in the fact that mathematics is not a separate discipline but part of a stream of ideas embracing its fields of application.

PAUL D. MINTON
Southern Methodist University

* Each member of the Association may buy one copy at a special price from H. M. Gehman, Mathematical Association of America, University of Buffalo, Buffalo 14, N. Y.

Analytic Functions. By R. Nevanlinna, H. Behnke and H. Grauert, L. V. Ahlfors, D. C. Spencer, L. Bers, K. Kodaira, M. Heins, J. A. Jenkins. Princeton Mathematical Series, no. 24. Princeton University Press, Princeton, 1960. vii+197 pp. $5.00.

This volume contains the principal addresses presented at a conference held in 1957 at the Institute for Advanced Study. R. Nevanlinna's address contains a new proof of Liouville's theorem on conformal mappings in R^n ($n>2$). Behnke and Grauert give a very readable introduction to and survey of the theory of analytic functions of several complex variables. The addresses of Ahlfors, Spencer, and Kodaira deal with problems in various complex analytic structures. Bers reports on quasiconformal mappings, Heins on conformal mapping of Riemann surfaces, and Jenkins on coefficients of univalent functions. While the addresses are mainly directed to experts, several, in particular Behnke and Grauert's, will be of interest to the general mathematical reader.

EDWIN HEWITT
University of Washington

An Introduction to Mathematical Statistics. By H. D. Brunk. Ginn, Boston, 1960. xi+403. $7.00.

The heart of this book is intended for a semester course in the beginning of mathematical statistics for advanced undergraduates and beginning graduates. There is an abundance of "starred" material permitting expanded use into a one-year course. The book seems very flexible, not only in being useful for either a one-half year or a full year course, but also in terms of its potential audience. It seems equally suited for students in subject matter fields who want a firm mathematical foundation in statistical procedures, as well as for students who are beginning a serious study of mathematical statistics. Probability foundations and random variables are presented in a fashion which is clear and sufficiently careful for this level. The development of statistical theory well reflects the modern developments of the last several decades. The book is well-written and has many worthwhile problems.

MEYER DWASS
Northwestern University

Linear Programming and Economic Analysis. By R. Dorfman, P. A. Samuelson, and R. M. Solow. McGraw-Hill, New York, 1958. ix + 527 pp. $10.00.

The theory of linear programming centers about a single problem: "Maximize the linear form $L(x)=\sum_{i=1}^{n} b_i x_i$ subject to the constraints $\sum_{j=1}^{n} a_{ij}x_j \leqq c_i$, $i=1, \cdots, M$." The question of existence is simple, and the problem is trivial if N and M are both small. The crux of the problem is that of determining feasible computational algorithms when M and N are large, of the order of several hundred or several thousand. Of some interest is also the problem of determining the analytic structure of the solution when the matrix $A=(a_{ij})$ has a particular form.

Although questions of this nature, as part of the general theory of linear inequalities, date back fifty to seventy-five years, they have only been intensively studied in the last ten years or so. This is a consequence of the great need for their solutions in the field of mathematical economics, in connection with input-output analysis and game theory, and in operations research in connection with allocation of resources, transportation theory, and scheduling theory.

The authors present a detailed account of the theory of linear programming and its economic background with discussion of numerous examples. The writing is leisurely and lucid, and a model of what good scientific writing should be.

The only drawback to the book lies in its treatment of dynamic processes. Although the methods of linear programming are not particularly applicable to this area, the authors have not included a discussion of alternative methods.

The book is thoroughly to be recommended to research workers in the fields of mathematical economics, game theory and operations research, to students entering these expanding fields, and to mathematicians who wish to obtain an overall view of this modern sphere of intellectual activity.

RICHARD BELLMAN
The RAND Corporation

College Algebra and Trigonometry. By William L. Hart. Heath, Boston, 1959. ix+466 pp. $5.50.

The text covers the traditional topics of college algebra and trigonometry, with more than the usual amount of analytic geometry. Graded problem lists are more than adequate. Topics are so arranged and compartmented that the text may easily be adapted to courses of various types and lengths.

Basic concepts of *variable*, *constant*, *function*, etc. are dealt with in traditionally ambiguous language. At one point *variable* is properly defined, but "x" is generally referred to as a "literal number," "letter," "number symbol," "general number." *Function* is initially defined in extension as a set of ordered pairs, etc., but is almost immediately identified with the independent variable, with function value, with "general value," and confused with equations or forms such as "x^2+3x," which, incidentally, are called "number expressions." The notion that a function has a domain and a range comes through rather better, yet the function concept does not assume a proper role throughout. It is not exploited, for instance, in treatment of sequences or "progressions."

Except for a few modern touches (*e.g.* section on probability), the text is a good traditional algebra-trigonometry with traditional virtues and vices. The conscientious student will emerge well drilled in techniques and in solving completely formulated problems. He may still not suspect that algebra has structure. He will probably not have improved his ability to construct algebraic proofs or to distinguish proof from heuristic.

ROBERT M. EXNER
Syracuse University

BRIEF MENTION

Regression Analysis. By E. J. Williams. Wiley, New York, 1959. ix+214 pp. $7.50.

This is not a theory book and not a mathematical book, but is based upon the author's years of consulting experience in applied statistics for experimental scientists.

Methods of Correlation and Regression Analysis. (3rd ed.) By Mordecai Ezekiel and Karl A. Fox. Wiley, New York, 1959, xv+548 pp. $10.95.

Another nonmathematical book on regression analysis; this time as an extension of an earlier book on correlation by Ezekiel. Mathematicians will do well to have a nonmathematical treatment of regression analysis on hand for loaning purposes.

NEWS AND NOTICES

EDITED BY LLOYD J. MONTZINGO, JR., University of Buffalo.

Readers are invited to contribute to the general interest of this department by sending news items to L. J. Montzingo, Jr., University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Professor A. A. Albert, University of Chicago, has been appointed to the Eliakim Hastings Moore Distinguished Service Professorship of the University.

Professor Howard Eves, University of Maine, represented the Association at the inauguration of Dr. Robert E. L. Strider, II, as President of Colby College on October 11, 1960.

Professor E. E. Moise, University of Michigan, has been appointed James Bryant Conant Professor of education and mathematics at Harvard University.

The Reverend R. J. Swords, S. J., was inaugurated as President of the College of the Holy Cross on June 16, 1960.

Arizona State University: Mr. R. W. Sanders, Lockheed Aircraft Corporation, has been appointed Assistant Professor; Associate Professor B. M. Ingersoll, Lamar State College of Technology, has been appointed Associate Professor; Associate Professor Simon Green, Assumption University of Windsor, Ontario, Canada, has been appointed Professor.

Vanderbilt University: Associate Professor G. F. Clanton, on leave from Baylor University as Lecturer at the University of Minnesota, has been appointed Assistant Professor; Assistant Professors B. F. Bryant and J. R. Wesson have been promoted to Associate Professors.

Washington University: Professor K. A. Hirsch, University of London, England, has been appointed Visiting Professor; Professor Helmut Schiek, University of Bonn, Germany, has been appointed Visiting Assistant Professor; Dr. F. J. Schnitzer, Wayne State University, has been appointed Research Associate.

Dr. R. D. Adams, University of Minnesota, has been appointed Assistant Professor at the University of Kansas.

Dr. R. E. Barlow, Stanford University, has accepted a position with the Communications Research Division of the Institute for Defense Analyses at Princeton University.

Dr. Archie Blake, Bendix Aviation Corporation, Ann Arbor, Michigan, has accepted a position as Manager, Analysis Section of the Raytheon Company, Sudbury, Massachusetts.

Dr. Eugene Butkov, McGill University, has been appointed Assistant Professor at St. John's University, Jamaica, New York.

Associate Professor M. L. Coffman, Abilene Christian College, has accepted a position as Senior Physicist with the Hamilton Standard Division of United Aircraft Corporation, Windsor Locks, Connecticut.

Professor A. J. Coleman, University of Toronto, has been appointed Head of the Department of Mathematics at Queen's University, Kingston, Ontario, Canada.

Mr. W. W. Collier, Harvard University, has accepted a position as Programmer for International Business Machines, Poughkeepsie, New York.

Mr. C. J. Cook, Purdue University, has been appointed Assistant Professor and Head of the Mathematics Department at Highpoint College.

Associate Professor William Craig, Pennsylvania State University, has been appointed Visiting Professor at the University of California, Berkeley.

Professor Wayne Dancer, University of Toledo, has been appointed Professor and Chairman of the Department of Mathematics and Natural Sciences at the Alaska Methodist University.

Mr. H. C. Dixon, Jr., Trinity College, has been appointed Head of the Mathematics Department of Berwick Academy, South Berwick, Maine.

Mr. J. W. Dutton, University of Kansas, has been appointed Teacher of Mathematics at the San Juan Unified School District, Carmichael, California.

Dr. C. W. Foard, Eastman Kodak Company, Rochester, New York, has been appointed Professor at the Rochester Institute of Technology.

Miss Phyllis R. Gansz, Baldwin School, Bryn Mawr, Pennsylvania, has been appointed Assistant Director of Admissions at Wilson College.

Professor P. C. Hammer, Director of the Numerical Analysis Laboratory of the University of Wisconsin, has returned from spending the academic year 1959–60 on a research leave in Zurich, Switzerland.

Professor Duncan Harkin, U. S. Naval Academy, has been appointed Professor and Chairman of the Department of Mathematics and Statistics at the American University, Washington, D. C.

Mr. K. P. Howe, University of Illinois, has accepted a position as Actuary Trainee with the New York Life Insurance Company, New York, New York.

Mr. Leonard Klafter, Army Map Service and the Department of the Air Force, has accepted a position as Head of Scientific Programming at the Bureau of Naval Weapons, Washington, D. C.

Mr. K. S. Kretschmer, Imperial Oil Limited, Toronto, Ontario, Canada, has accepted a position as Planning Associate with the Socony Mobil Oil Company, Incorporated, New York, New York.

Mr. C. J. Labovites, Headquarters, United States Air Force, has accepted a position with the Bureau of Naval Weapons, Washington, D. C.

Mr. J. A. Marlin, Assistant Instructor, University of Missouri, has accepted a position as a member of the Technical Staff of the Bell Telephone Laboratory, Winston-Salem, North Carolina.

Mr. J. W. Milsom, Southwestern Louisiana Institute, has accepted a position as Instructor at Texas College of Arts and Industries.

Mr. D. L. Muench, McQuaid Jesuit High School, Rochester, New York, has been appointed Assistant Professor at the United States Naval Academy.

Dr. D. E. Myers, Millikin University, has been appointed Assistant Professor at the University of Arizona.

Mr. P. T. Nugent, Assistant Instructor, Miami University, has been appointed Instructor at Vincennes University.

Mr. H. E. Pickett, Assistant Instructor, University of California, Berkeley, has accepted a position as Research Mathematician in Analytical and Physical Measurements, Richmond Laboratory of the California Research Corporation.

Dr. R. F. Rinehart, Case Institute of Technology, has returned from a two-year leave of absence as Director of Special Research and Operations Research at Duke University.

Mr. G. R. Rising, Greece Olympia High School, Rochester, New York, has been appointed Mathematics Coordinator for the Norwalk Public Schools, Norwalk, Connecticut.

Mr. R. L. Sauder, Virginia Military Institute, has accepted a position as Programmer with the General Electric Computer Department, Phoenix, Arizona.

Associate Professor N. T. Seeley, Jr., Agricultural and Technical College of North Carolina, has been appointed Assistant Professor at Morgan State College.

Associate Professor R. L. Shively has been awarded a leave of absence from Western Reserve University to accept a National Science Foundation Faculty Fellowship for study at the University of California, Berkeley.

Mr. J. R. Singletary, Fishburne Military School, Waynesboro, Virginia, has been appointed Master in Field of Mathematics at Perkiomen School, Pennsburg, Pennsylvania.

Miss Ann A. Steinbright, Eastern Baptist College, has accepted a position as Mathematician-consultant with the E. I. DuPont de Nemours and Company, Wilmington, Delaware.

Mr. R. F. Steinen, United States Rubber and Development Research Center, Wayne Township, New Jersey, has been appointed Head of the Mathematics Department of Hawken School, Lyndhurst, Ohio.

Mr. R. G. Tobey, Harvard University, has been appointed Instructor at the College of Wooster.

Mr. C. B. H. Watson, Canada Life Insurance Company, Toronto, Ontario, Canada, has accepted a position as Actuary with the Wyatt Company, Consulting Actuaries, Washington, D. C.

Associate Professor J. C. Wilson, Central College, Pella, Iowa, has been appointed Assistant Professor at Florida Presbyterian College.

Miss Edla G. Berger, Mathematician, Equitable Life Assurance Society, died June 17, 1960. She was a charter member of the Association.

Dr. R. P. Peterson, Jr., Senior Mathematician, Burroughs Electrodata, died June 19, 1960. He was a member of the Association for fourteen years.

Mr. George Tipton, Sr., Head of the Mathematics Department, Henderson County Junior College, died May 6, 1960. He was a member of the Association for twelve years.

Professor Oswald Veblen, Institute for Advanced Study, died August 11, 1960. He was a Charter Member of the Association.

Professor G. A. William, Oregon State College, died April 22, 1960. He was a member of the Association for thirty-five years.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor Henry L. Alder, Secretary, announces that the following 87 persons have been elected to membership by the Board of Governors on applications duly certified.

NED PAUL ANDERSON, Student, Euclid Senior High School, Ohio

RUDOLPH F. APPEL, Programmer, Dime Savings Bank, Brooklyn, New York

ENRIQUE BAYÓ, M.A. (Columbia) Asso. Prof., University of Puerto Rico

MRS. LYDIA T. BILLS, A.B. (Wheaton Coll.) Teacher, Windsor Mountain School, Lenox, Massachusetts

DENNIS K BOSWELL, B.A. (Vanderbilt) Arcadia, California

JAMES O. BROOKS, M.A. (Michigan) Asst. Prof., Haverford College

SARAH BROOKS, B.A. (Albany State) Teacher, Utica, New York

ROBERT V. BUDNY, Student, Massachusetts Institute of Technology

RONALD C. BZOCH, Ph.D. (Illinois I.T.) Asst. Prof., University of Utah

ALICE M. CALHOUN, B.A. (Illinois) Programmer-Analyst, General Telephone and Electronic Labs., Bayside, New York

DAVID M. CARGILL, M.Ed. (Boston) Head of Dept., Dean Junior College

ARNOLD M. CHANDLER, M.S. (Wisconsin) Supervisor, Department of Public Instruction, Madison, Wisconsin

JOHN D. CLOUD, B.S. (Alabama) Grad. Asst., University of Alabama

ROBERT K. COBB, Student, Southwest Missouri State College

ROBERT E. COOMBS, B.S. (U.S. Naval Academy) Instr., Lakeside School, Seattle, Washington

ROBERT CROTEAU, M.Sc. (Montreal) Asst. Prof., Laval University

GEORGE S. CUNNINGHAM, M.Ed. (Maine) Director of Math. Education, New Hampshire State Department of Education

LEROY M. DAMEWOOD, M.S. (Oregon Coll.) Instr., Bowdoin College

GABRIEL F. DELVILLAR, M.S. (Mexico) Prof., University of the State of Lower California

FRANK R. DEMEYER, Student, Seattle University

DANIEL G. DEWEY, M.A. (Kansas) Asst. Prof., College of the Holy Cross

HENRY H. DIEHL, M.A. (Ohio) Asst. Prof., Wittenberg University

L. M. EVANS, M.S. (Texas State) Teacher, Highland Park High School, Dallas, Texas

ARTHUR G. EYLES, B.Eng. (McGill) Chief Computer Engr., Analdata Engineering, Boston, Massachusetts

SARAH A. FLUCK, Student, Temple University

ARNE HARTWICK GARNESS, M.S. (Wisconsin) Instr., Concordia College

JOHN W. GAROFALO, B.S. (St. Bonaventure) Programmer Math., Thiokol Chemical Corp., Elkton, Maryland

JOSEPH GELLER, B.A. (Montclair Teacher, Central High School Newark, New Jersey

VIOLA GRIFFIN, M.S. (Texas State) Teacher, Queen City Independent Schools, Texas

LOUIS J. GRIMM, M.S. (Georgia I.T.) Chemist, U.S. Public Health Service, Savannah, Georgia

LOUIS N. GROSS, B.S. (Chicago) Mathematician, Mitre Corporation, Lexington, Massachusetts

HARRY P. HALE, M.S. (Pittsburgh) Asso. Prof., Wayne State University

THOMAS E. HANRAHAN, M.S. (DePaul) Research Asst., American Medical Association, Chicago, Illinois

HAROLD HENDLER, A.B. (Brooklyn Coll.) Teacher, Andrew Jackson High School, Canbria Heights, New York

ROBERT F. HENKEL, Technical Analyst, McDonnell Aircraft Corporation, St. Louis, Missouri

NEIL W. HENRY, M.A. (Dartmouth Coll.) Research Asst., Sociology, Columbia University

EARL W. HESSEE, B.N.E. (N. Carolina) Manager, Lockheed Aircraft Corporation, Dawsonville, Georgia

THOMAS P. HETTMANSPERGER, Student, Indiana University

KONRAD J. HEUVERS, Student, Stanford University

ROGER B. HOOPER, M.A. (Wesleyan) Head of Dept., Trinity-Pawling School, Pawling, New York

FRANK L. JENKINS, JR., M.A. (Illinois) Instr., Fenn College

MIKE KAPLAN, Student, Hunter College

MOSES S. KARMAN, M.A. (Brooklyn Coll.) Head of Dept., Ramaz Upper School, New York, New York

RUBIE L. KELLEY, B.S.E. (Arkansas) Teacher, Monahans High School, Texas

BRUCE W. KING, B.S. (Albany State) Teacher, Burnt Hills-Ballston Lake High School, Burnt Hills, New York

SAMUEL W. KODIS, A.B. (California) Instr., San Diego State College

CLARENCE R. KROPP, B.S. (Great Falls Coll.) Teacher, Paris Gibson School, Great Falls, Montana

DARRYL E. KUHNS, Student, Miami University

DONALD L. LANG, B.S. (Great Falls Coll.) Teacher, Geraldine High School, Montana

PETER LEES, B.S. (New Zealand) Asst. Master, OneLunga High School, Auckland, New Zealand

HILBERT LEVITZ, B.A. (N. Carolina) Grad. Asst., Pennsylvania State University

PHILIP L. MATTSON, B.S. (California State) Computer-Programmer, Curtis Publishing Company, Philadelphia, Pennsylvania

VERNON A. MAZEMKE, B.A. (Wartburg Coll.) Grad. Student, Wartburg College

CARL M. METZLER, B.S. (Goshen Coll.) Instr., Goshen College

MRS. EMMA F. MIXON, M.S. (Louisiana State) Asst. Prof., Northeast State College

JACK D. MUNN, M.A. (Alabama) Asst. Prog., Mississippi Southern College

SAM NEWMAN, B.S. (Columbia) Mathematician, NAFEC, Atlantic City, New Jersey

HELEN O'DEA, M.A. (California) Asso. Prof., Northern Montana College

MARGARET A. PAPARILLO, M.A. (Columbia) Teacher, West Hempstead Junior-Senior High School, New York

BRUCE B. PETERSON, M.A. (Syracuse) Grad. Asst., Syracuse University

ROBERT F. PIERCE, Ed.M. (Boston) Editor, D. C. Heath & Company, Boston, Massachusetts

ELAINE E. POSTELNECK, B.A. (Barnard) Teacher, Sanford H. Calhoun High School, Merrick, New York

FREDERICK RENVYLE, A.B. (Harvard) Teacher, Abington Senior High School, Massachusetts

JOAN RICHARDSON, B.S. (Minnesota) Instr., Arizona State University

ROY M. ROBERTS, M.S. (Tennessee) Teacher, Maryville High School, Tennessee

ALLEN I. RUBENSTEIN, Student, College of the City of New York

JOHN M. SAADE, A.B. (Emory) Grad. Student, Emory University

THOMAS L. SAINDON, Student, Illinois Institute of Technology

HENRY J. SCHAFER, M.Ed. (Puget Sound) Teacher, Wilson High School, Tacoma, Washington

RALPH W. SCHREIBER, B.S. (Kansas State Coll.) Teacher, Harrison High School, Colorado Springs, Colorado

GEORGE S. SHAPIRO, Student, Forest Hills High School, New York

KENNETH L. SISSERSON, M.A. (Southern Methodist) Teacher, Monnig Junior High School, Fort Worth, Texas

RONALD J. SRODAWA, Student, Sewitt High School, Detroit, Michigan

MRS. VELLA STEWART, B.S. (Southwestern) Teacher, Cyril High School, Oklahoma

JOSEPH T. SUCHAR, B.S. (Kutztown State Coll.) Instr., Rider College

THURMAN L. TALLEY, M.S. (N. Mexico) Grad. Student, Florida State University
STEPHEN J. TEMPERO, Student, University of Nebraska
MICHAEL T. THAYER, Student, Walnut Hills High School, Cincinnati, Ohio
ROSEMARY C. TIGHE, M.A. (Hunter Coll.) Teacher, Forest Hills High School, New York
GARY G. TRESSEL, B.S. (Great Falls Coll.) Teacher, Granite County High School, Philipsburg, Montana
JEAN C. TURGEON, Ph.D. (Columbia) Lecturer, Sir George Williams University
MANUEL TAN UY, Student, Massachusetts Institute of Technology
ROBERT H. WALKER, M.E. (N. Dakota) Teacher, Lewis and Clark Junior High School, Billings, Montana
KENNETH L. WHIPKEY, M.A. (Kent State) Instr., Youngstown University
MRS. JOYCE W. WILLIAMS, Ph.D. (Illinois) Boxboro Depot Road, Littleton, Massachusetts
ROBERT E. WILLIAMS, B.S. (Missouri) Grad. Asst., Missouri University
P. K. WONG, M.S. (Carnegie I.T.) Instr., Carnegie Institute of Technology

THE FORTY-FIRST SUMMER MEETING OF THE ASSOCIATION

The Forty-first Summer Meeting of the Mathematical Association of America was held at Michigan State University, East Lansing, Michigan, from Monday, August 29 through Wednesday, August 31, 1960 in conjunction with summer meetings of the American Mathematical Society, the Society for Industrial and Applied Mathematics, the Pi Mu Epsilon Fraternity, and Mu Alpha Theta. There were registered 760 persons, including 568 members of the Association.

Sessions of the MAA were held on Monday morning and afternoon, on Tuesday morning and on Wednesday afternoon. All sessions were held in Anthony Hall Auditorium of Michigan State University. Presiding officers at the three Earle Raymond Hedrick Lectures were President C. B. Allendoerfer, First Vice-President A. S. Householder, and Second Vice-President Harley Flanders, at the session on special programs in the teaching of undergraduate mathematics Professor R. J. Wisner, at the session on the role of abstract and concrete approaches in the teaching of mathematics Professor J. C. Oxtoby, at the business meeting and lecture on Tuesday morning President Allendoerfer, and at the lecture on Wednesday afternoon Professor J. G. Herriot. The ninth series of Earle Raymond Hedrick Lectures were delivered by Professor Ivan Niven of the University of Oregon. The Program Committee for the meeting consisted of J. C. Oxtoby, Chairman; W. E. Briggs, J. G. Herriot, K. O. May, R. J. Wisner.

FIRST SESSION OF THE ASSOCIATION

The Earle Raymond Hedrick Lectures: *Some Aspects of Diophantine Approximation*, Lecture I, by Professor Ivan Niven, University of Oregon.

The Hedrick Lectures by Professor Niven will be published by the Association at an early date.

Session on Special Programs in the Teaching of Undergraduate Mathematics

The Honors Program at Princeton, by Professor Albert W. Tucker, Princeton University.

A freshman honors course, started ten years ago by Emil Artin, has been gradually extended through sophomore and junior years (see Artin, *Calculus and Analytic Geometry* (C.U.P. 1957), M.A.A., Buffalo, also Nickerson, Spencer and Steenrod, *Advanced Calculus*, Van Nostrand, 1959). Only outstanding applicants, liberal arts or engineering, are accepted initially (top ten percent of freshman class, roughly) after careful screening via school records, CEEB scores, and interviews. Those with "advanced placement" calculus present a special problem; most go into an advanced division of freshman honors, a few into sophomore honors. Transfers into the honors program are rare, but transfers out occur at frequent intervals on instructor's advice or student's request. Each student is responsible for routine exercises on his own. Thirty of this year's forty-five mathematics majors, juniors and seniors, are products of the freshman-sophomore honors courses. The program demands much from students and staff, but is highly prized by both.

The New Program at Wesleyan, by Professor Robert A. Rosenbaum, Wesleyan University.

The College of Quantitative Studies is one of several experimental programs designed to minimize academic book-keeping, to emphasize independent work at an early stage of the student's career, and to encourage greater responsibility in the student for his own education. With its focus in the mathematics department, the College also draws on many other departments—its Directors of Studies now come from astronomy, economics, and political science as well as from mathematics. These Directors have the responsibility of supervising the students' choice of programs, of assessing satisfactory progress (in lieu of grades), of arranging for qualifying and comprehensive examinations, set by outside examiners, and of recommending candidates for the B.A.

Students enter the program at the beginning of the sophomore year. From the start, about one-quarter of their work is devoted to problems and projects, of a non-textbook sort. Some of the problems have an academic flavor, originating in such departments as physics, biology, and psychology; others come from business, industry, and government. It should be emphasized that some of the problems come from within mathematics itelf; and it is hoped to produce more future mathematicians, as well as graduates who can use mathematics effectively, than Wesleyan has produced in the past.

The Program at Dartmouth, by Professor John G. Kemeny, Dartmouth College.

After seven years of planning and experimentation, the new Dartmouth mathematics program is in operation. The department has attempted to strike a healthy balance between courses designed for the prospective mathematician, for the average student, and for men interested in the physical, biological and social sciences.

In the first two years the students may be placed into an honors program (somewhat similar to that described by A. W. Tucker), a regular sequence, or a special sequence for the non-physical-scientist. The former consists of a year of calculus followed by a course in logic-sets-probability-linear algebra, which is used for the introduction of functions of several variables. The special sequence combines introduction to the basic ideas of calculus with the more modern concepts mentioned. A lecture system has been employed successfully in the non-honors sections.

Upper class courses again correspond to the three tracks. Dartmouth's unusual three-term three-course system allows a maximum concentration on mathematics courses, with the possibility of completing a three-semester sequence in one year. This means that honors majors reach graduate courses in record time.

The liberal use of undergraduate research assistants both in the computation center and as personal assistants to staff members has resulted in a high degree of undergraduate interest in productive mathematical activity.

The Undergraduate Thesis Program at Reed, by Professor Lloyd B. Williams, Reed College.

The presentation of a thesis, written during the senior year is a requirement for the B.A. degree at Reed College. Practices vary somewhat from department to department within the College in administering this requirement, but generally the thesis is ". . . an independent project of research or of critical or creative work . . ." (quoted from the College Catalog).

The thesis requirement has been in effect since the first graduating class of the College in 1915. Since then, two hundred theses have been written by students majoring in mathematics. Professor F. L. Griffin has described the experience with this program in three articles in this MONTHLY, vol. 37, 1930, vol. 49, 1942, and vol. 58, 1951.

This paper considers trends in the program during the last ten years.

The Philips Visitor Program at Haverford, by Professor James O. Brooks, Haverford College.

The Philips Visitor program at Haverford College affords each year a one semester course in

some advanced undergraduate subject given by a leading mathematician or mathematicians of the country. This course makes an outstanding supplement to regular course offerings of the Haverford mathematics department. Students benefit not only from exposure to certain subject matter but also from contact with distinguished scientists and teachers.

The program began at Haverford with a course of lectures on the calculus of variations by Professor Richard Courant. Other Visitors appearing since in mathematics have included Professors Ross Beaumont, Arthur Coble, Charles Coulson, Ralph Fox, Paul Halmos, Mark Kac, Deane Montgomery, F. D. Murnaghan, John Oxtoby, Hans Rademacher, James Stoker, Robert Thrall, and Albert Tucker. This coming year Professors Marston Morse and Andre Weil will each give a series of lectures.

The program is endowed by a gift of a Haverford alumnus, William Pyle Philips.

Haverford students, especially math majors, take the Philips lectures plus an additional recitation session as a three semester hour course. The students write up notes on the lectures. The lectures will be published by Ginn and Co. in a series entitled the "Haverford Mathematical Monographs."

SECOND SESSION OF THE ASSOCIATION

Hedrick Lecture II, by Professor Niven.

Session on the Role of Abstract and Concrete Approaches in the Teaching of Mathematics

Toward the Abstract: The Problem of Communications of Ideas, by Professor Arnold E. Ross, University of Notre Dame.

A continual and a rapid change has become the way of life of this generation and of uncountable generations to come.

Among the intellectual skills used in both science and technology to probe into the unknown, the art of "theorizing" is probably the most far reaching. Thus we are compelled by considerations of survival to bring the most powerful tool of exploration, viz., abstract thinking, within the grasp of our young. Since attitudes are now as vital as skills, now more than ever education must seek to develop intense curiosity, keenness of observation, a capacity for invention, and many other oft-referred-to virtues of the pioneering spirit.

The problem of communication of ideas, in particular of mathematical ideas, is one of the key questions before us. I propose that we consider this problem and that in doing this we subject to a painful reappraisal many of the ways in which we now use language as a tool of education.

What Price Abstraction? by Professor Mark Kac, Cornell University.

This paper will appear in this MONTHLY.

THIRD SESSION OF THE ASSOCIATION

Hedrick Lecture III, by Professor Niven.

Business Meeting of the Association.

Report of the Committee on the Undergraduate Program in Mathematics, by Professor R. Creighton Buck, University of Wisconsin.

Progress reports of the work of this Committee will appear from time to time in this MONTHLY. The first such report, namely that of the Teacher-Training Panel, will appear in the December issue.

Lecture: *Convergence Regions for Continued Fractions and Certain Other Infinite Processes*, by Professor Wolfgang J. Thron, University of Colorado.

The iterative definition of continued fractions is used as a starting point. Let $t_n(z) = a_n/(b_n+z)$ and let $T_n(z) = T_{n-1}(t_n(z))$, $T_1(z) = t_1(z)$. Then the continued fraction is defined as the sequence $\{T_n(0)\}$. Convergence criteria for this and for related infinite processes, where $t_n(z)$ is not neces-

sarily a linear fractional transformation, are derived. The criteria are of the convergence region type, that is a region G is determined such that if all $a_n \in G$ (it is now assumed that t_n depends on one parameter only which is called a_n) then the sequence $\{T_n(0)\}$ converges. The methods are based on a determination of value regions and corresponding element regions. Various methods of proof are discussed. Results can be obtained by using the Stieltjes-Vitali theorem or in certain cases by elementary methods involving a sequence of "nested sets" and an estimate of their diameters.

FOURTH SESSION OF THE ASSOCIATION

Lecture: *Fourier Transforms*, by Professor Jacob Korevaar, University of Wisconsin.

The Fourier transform $g = Tf$ is defined by the formula

$$g(x) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} f(u) e^{-ixu} du,$$

the conjugate Fourier transform $h = \overline{T}f$ by the analogous formula obtained on replacing $-i$ by i. The inversion theorem "if $g = Tf$ then $f = \overline{T}g$" and the derivative rule $TDf = \{ix\}Tf$ make the Fourier transformation an important tool of applied mathematics. Unfortunately the classical theory places severe restrictions on the growth of f as $x \to \pm\infty$. It is shown in this lecture how the theory of pansions makes it possible to use the Fourier transformation even when the functions involved grow as fast as $\exp(cx^2)$. Among the applications is a uniqueness theorem for the heat equation.

A pansion is a formal series $\sum c_k v_k$ of normalized Hermite functions. Pansions which are Hermite expansions of functions or distributions are identified with those functions or distributions. Within the class of pansions one can form the Fourier transform, the conjugate Fourier transform, the derivative, the product by $\{x\}$, etc. The definitions given agree with the classical definitions whenever the latter apply. For example, the Fourier transform of the pansion $\sum c_k v_k$ is the pansion $(\sum c_k Tv_k =) \sum (-i)^k c_k v_k$. The inversion and derivative rules now become valid without restriction. For details see Trans. Amer. Math. Soc., vol. 91, 1959, pp. 53–101.

SPECIAL SESSIONS OF THE ASSOCIATION

On Tuesday afternoon at 3:15 an open conference on High School Contests was held in Room 120, Physics-Mathematics Building, with 68 persons present. President Allendoerfer presided. The question of whether problems on "modern mathematics" should be included in future contests received particular attention. There was agreement that an attempt should be made to introduce such problems into the contest, but that not more than 5 out of 40 problems should be of this type in next year's contest.

On Wednesday evening at 7:15 a sound film in color on "Mathematical Induction" with Leon A. Henkin as lecturer and produced by the MAA's Committee on Production of Films was shown in Anthony Hall Auditorium. The film was made by Palmer Films, San Francisco, and consists of two parts, each 30 minutes long.

MEETING OF THE BOARD OF GOVERNORS

The Board of Governors of the Association met on Monday evening in Room 221, Physics-Mathematics Building of Michigan State University with thirty-two members present.

Professor W. S. Loud of the University of Minnesota was elected to the Board of Governors to fill the balance of the term of Professor J. M. H. Olmsted who has moved from the Minnesota Section.

Professor R. P. Bailey of the Naval Academy was elected to the Board of Governors to fill the balance of the term of Professor R. C. Yates who has moved from the Maryland-D.C.-Virginia Section.

The Board voted to invite Professor R. H. Bing of the University of Wisconsin to

deliver the tenth series of Earle Raymond Hedrick Lectures at the 1961 Summer Meeting.

The Board approved the following schedule of future meetings: Hotel Willard, Washington, D. C., January, 1961; Oklahoma State University, August 1961; Sheraton-Gibson Hotel, Cincinnati, Ohio, January 1962; University of British Columbia, August 1962; University of California, Berkeley, January 1963; University of Colorado, August 1963; University of Michigan, August 1964; Rutgers-The State University, New Brunswick, New Jersey, August 1966.

The Board acting in accordance with Article III, Section 5, of the By-Laws of the Association appointed Professor Harry M. Gehman Executive Director as of September 1, 1960, under terms and conditions of employment fixed by the Finance Committee.

The Board acting upon the recommendations of its Committee on Publications under the chairmanship of Professor Roy Dubisch approved the following:

I. The present Carus and Slaught Committees shall be abolished and, in their place, there shall be established a standing Committee on Publications. This committee shall have over-all responsibility for all publications of the Association including the MONTHLY, Carus, Slaught, and CUP publications. The editor of the MONTHLY shall be an *ex officio* member of the committee.

II. Two new publishing ventures shall be initiated as soon as possible.

A. The Association shall take over the publication and editorship of the *Mathematics Magazine* to be sold on a subscription basis with, possibly, a reduced rate to members of the Association and the National Council of Teachers of Mathematics. The mathematical level of the *Magazine* shall be below that of the MONTHLY but above that of the *Mathematics Teacher*. The editor of the *Magazine* shall be an *ex officio* member of the Committee on Publications.

B. A new series of publications, tentatively titled *MAA Studies in Mathematics*, shall be initiated to be published, if possible, in cooperation with a commercial publisher (as is now done for Carus) or, alternately, to be published by the American Mathematical Society. The first two volumes would be collections of papers, some reprints of older MONTHLY articles and some new manuscripts, of top quality exposition which have recently been collected by Professor R. V. Andree.

III. An editorial office shall be set up in Buffalo as soon as possible to handle the preparation of manuscripts for the printer, proof reading, reprint orders, etc.

IV. The publication of lists of new members in the MONTHLY shall be discontinued effective January 1961.

BUSINESS MEETING OF THE ASSOCIATION

A business meeting of the Association was held on Tuesday morning with President Allendoerfer presiding. The Secretary reported that the membership of the Association was 9913, an increase of 15% since the corresponding date last year.

The Secretary then reported on the actions taken by the Board of Governors the previous evening and introduced Professor Harry M. Gehman as Executive Director whose appointment to the new position was received with great applause by the audience.

MEETING OF SECTION OFFICERS

A meeting of representatives of the Sections of the Association was held on Tuesday evening in Room 221, Physics-Mathematics Building. Approximately 48 persons were present representing 25 of the 27 Sections of the Association.

Professor Russel C. Phelps of the National Science Foundation reported on N.S.F. Programs for the support of science education projects. Professor J. G. Kemeny gave a report of the Panel on Teacher Training of the Committee on the Undergraduate Pro-

gram in Mathematics. Professor Roy Dubisch discussed the Reports of Section Meetings in the MONTHLY. The Section representatives voted in favor of continuing to print abstracts of talks delivered at section meetings, but on a voluntary basis, i.e., abstracts would not be demanded from speakers who prefer not to have them printed. A discussion of the "Role of Industrial Members in the Association" was opened by brief talks by Dr. H. O. Pollak of Bell Telephone Company and Dr. B. H. Mount, Jr., of Westinghouse Electric Corporation.

MEETINGS OF OTHER ORGANIZATIONS

The American Mathematical Society held its sessions from Tuesday afternoon through Friday. The colloquium speaker was Professor S. S. Chern of the University of California, Berkeley, who spoke on "Geometrical Structures on Manifolds," and invited addresses were given by Professor Paul Halmos of the University of Chicago on "Recent Progress in Ergodic Theory" and Professor P. E. Conner of the University of Virginia on "Involutions and Equivariant Maps."

The Society for Industrial and Applied Mathematics had three sessions. The first, at which the von Neumann Lecture was presented by Professor L. V. Ahlfors of Harvard University, was held Wednesday afternoon at 1:30, the second Thursday evening at 8:00, and the third Friday afternoon at 2:40.

The Pi Mu Epsilon Fraternity held a luncheon and business meeting Tuesday noon. A session for seven 20-minute papers by students was held beginning at 2:45 P.M. Mu Alpha Theta, the National High School and Junior College Mathematics Club, held a luncheon meeting on Wednesday.

ARRANGEMENTS, ENTERTAINMENT, AND RECREATION

The Committee on Arrangements for the meeting consisted of W. E. Deskins, Chairman; H. L. Alder, J. S. Frame, F. Herzog, L. M. Kelly, M. L. Tomber, G. L. Walker, C. P. Wells, J. W. T. Youngs.

Registration headquarters was located in the lobby of Snyder Hall. Dormitory and cafeteria accommodations were provided by Michigan State University. The textbook exhibit and the Mathematical Sciences Employment Register were located in Snyder Hall near the registration desk.

On Monday evening there was an informal coffee hour at 8:00 P.M. in Phillips Hall, Lower Lounge. An informal tea was held Tuesday afternoon from 4:00 to 6:00 P.M. in Phillips Hall, Lower Lounge. A chicken barbecue was held Wednesday evening at 5:00 P.M. on the campus. The usual SIAM social evening was held on Thurdsay evening.

A resolution of thanks prepared by Professor W. T. Martin and adopted by the participating organizations expressed the most sincere appreciation to Michigan State University, to the local arrangements committee, and to all those on the University staff for the excellent arrangements which have been provided for the meeting. The resolution expressed the opinion of the participating organizations that the facilities provided at Michigan State have not been excelled within the memory of those attending, and that everyone will carry away the recollection of a most pleasant and profitable stay.

HENRY L. ALDER, *Secretary*

THE APRIL MEETING OF THE OKLAHOMA SECTION

The annual spring meeting of the Oklahoma section of the Mathematical Association of America was held at the University of Oklahoma, Norman, Oklahoma, on April 16, 1960. Professor Katherine Mires, Chairman of the Section, presided during the meeting, which was devoted to the reading of mathematical papers of both research and expository nature. There were 56 persons present, including 47 members of the Association.

The following invited addresses of roughly twenty minutes each were presented during the morning session.

1. *Some properties of the von Sterneck or Ramanujan function*, by Professor C. A. Nicol, University of Oklahoma.

If k and n are integers such that $k \geqq 0$, $n > 0$, then $\phi(n)\mu(n/(k, n)/\phi(n/(k, n))$ is called the von Sterneck or Ramunjan function and is denoted by $\Phi(k, n)$. Here (k, n) is the G.C.D. of k and n, ϕ is the Euler totient, and μ is the Moebius function. Some arithmetic properties of the von Sterneck function are discussed as well as its appearance in some generating functions.

2. *Radial plane flows of a nonlinear viscous fluid*, by Professor J. A. Nickel, Oklahoma City University.

The viscous fluid considered is an incompressible isotropic Reiner-Rivlin fluid having a nonlinear stress function of the form $T = -pI + mD + nD^2$, where m and n are analytic functions of the scalar invariants of the rate of displacement matrix D and the dimensional parameters of the flow. If m, a generalization of the classical viscosity, is assumed to be independent of temperature and pressure in the flow, it is shown that a necessary and sufficient condition of a radial flow to exist is that m is a constant, and consequently, the velocity field is identical with that of the classical flow. Furthermore, the normal pressures are no longer hydrostatic, but depend upon the coefficient s in the stress matrix.

3. *Effects of correlation upon a variance testing procedure*, by Dr. Frank McFeely, Montana State College, Visiting Scholar at the University of Oklahoma.

4. *On ideals in integral quaternions*, by Professor W. A. Rutledge, University of Tulsa.

Using the definition given by Hurwitz, the (noncommutative) integral domain of quaternions is a principal ideal domain J. Similarity is defined by: a is similar to b if there exists c such that $caJ = cJ \cap bJ$, with $cJ + bJ = J$. This relation is studied in terms of norm, bound (ideal) and minimal containing ideal.

Papers presented during the afternoon session were:

5. *Green rectangles*, by Professor Arthur Bernhart, University of Oklahoma.

The numerical solution of the Laplace difference equation in two dimensions suggests the study of arrays in which each entry is the average of its four neighbors. To obtain factors for arbitrary boundary conditions a singularity is introduced at an interior point analogous to the Green's function.

6. *A problem in transforming hyperbolic to rectangular coordinates*, by Professor W. B. Garrison, University of Tulsa.

This paper concerns a problem encountered in radio navigation systems, in particular the system called LORAN, that makes use of two families of hyperbolas for positioning purposes.

7. *Existence of surfaces in a certain conformal correspondence*, by Professor T. K. Pan, University of Oklahoma.

Let v and $\bar{v}$ be two corresponding vector fields associated with two surfaces S and $\overline{S}$ respectively. Let the indicatric torsions of v and $\bar{v}$ along two corresponding curves C and $\overline{C}$ be equal (T. K. Pan, Torsion of a vector field, Proc. Amer. Math. Soc., vol. 7, p. 453). The existence of such surfaces is investigated. It is found that the general solution depends on one arbitrary function of two variables.

8. *Filters and ultrafilters*, by Mr. J. E. Allen, Oklahoma State University.

The ordered set of filters on a set E is inductive, and thus the theorem of Zorn implies that for every filter on a set E, there exists an ultrafilter finer than it. This theorem is used to obtain the main result of this paper: Every filter is the intersection of all the ultrafilters finer than it.

9. *A medieval antecedent of Descartes' graphing technique*, by Dr. T. M. Smith, University of Oklahoma, introduced by the Secretary.

A crude technique for generating graphic figures appeared in 14th-century Europe as a consequence of scholastic preoccupation with "the intension and remission of forms" and served as an aid to the comprehension of logical analyses of motion; it became a part of the kinematic mathematical-physical tradition culminating in Galileo's delineation of the law of free fall in the 17th century.

10. *The Boole table generalized*, by Dr. W. E. Stuermann, University of Tulsa.

In an earlier item in the Classroom Notes of this MONTHLY, the writer reported a graphical method for analyzing Boolean functions which were displayed in conjunctive or disjunctive normal form. The present paper shows how the Boole table may be used to analyze any Boolean function which can be constructed from the operations of sum, product, and complement, no matter how deeply and complexly the variables are overlaid by stacks of these operations.

11. *Markov chains as random input automata*, by Mr. Allen Davis, University of Oklahoma. (This paper has been accepted for publication in this MONTHLY.)

R. V. ANDREE, *Secretary*

PROPOSED AMENDMENTS TO THE BY-LAWS OF THE M.A.A.

At the annual business meeting of the Association to be held at the Willard Hotel in Washington, D. C., on Thursday, January 26, 1961, motions will be made to amend the By-Laws as follows:

1. That Article III, Section 8 (c) be amended by adding these two sentences: "A Governor who has moved permanently from the Section by which he was elected shall be considered to have ended his term of office on the Board. If the Governor has moved from the Section because he is no longer employed there, it shall be interpreted that he has moved permanently from the Section."

2. That a new section, to be numbered Article III, Section 13, be inserted in the By-Laws to read as follows:

"(a) There shall be an Executive Director who shall be a paid employee of the Association. He shall have charge of the central office of the Association, and shall carry out such other duties as may be assigned to him by the Board. He shall be responsible to the Board, and shall attend meetings of the Board, the Executive Committee, and the Finance Committee, but he shall not be *ex officio* a member of these bodies.

(b) The Executive Director shall be elected by the Board under terms and conditions of employment fixed by the Finance Committee."

HENRY L. ALDER, *Secretary*

THE EMPLOYMENT REGISTER

The Mathematical Sciences Employment Register, established by the American Mathematical Society, the Mathematical Association of America, and the Society for Industrial and Applied Mathematics, will be maintained at the Annual Meeting at the Willard Hotel in Washington, D. C. on January 24, 25, and 26, 1961. The Register will be conducted from 9:00 A.M. to 5:00 P.M. on each of these three days.

The Employment Register Desk will be located in room 228. There is no charge for registering to either job applicants or to employers, except when the Late Registration Fee for employers is applicable. Provision will be made for anonymity of applicants upon request.

Job applicants and employers who wish to be listed will please write to the Employment Register, 190 Hope Street, Providence 6, Rhode Island, for application forms and for position description forms, which must be completed and returned to Providence not later than January 4, 1961, in order to be included free of charge in the listings at the

meeting in Washington, D. C. Forms which arrive after this closing date, but before January 17, will be included in the listings at the meeting for a Late Registration Fee of $3.00, and will also be included in the printed listings, but not until ten days after the meeting. The printed listings will be available for distribution both during and after the meeting. The prices are as follows: Position descriptions, $2.00; listing of applicants, academic only, $5.00; comprehensive listing of applicants, academic, industrial, and government, $20.00.

It is essential that applicants and employers register at the Employment Register Desk promptly upon arrival at the meeting to facilitate the arrangement of appointments.

CARUS MONOGRAPH NUMBER 5

The above Monograph entitled "History of Mathematics in America before 1900" by D. E. Smith and Jekuthiel Ginsburg was published in 1934. For many years it has been out of print.

Recently the office of the Association has received a number of inquiries from members wishing to purchase Monograph 5. Anyone who owns a copy of this Monograph and wishes to sell it may notify the Buffalo office of the MAA, and prospective purchasers will be notified.

CALENDAR OF FUTURE MEETINGS

Forty-fourth Annual Meeting, Willard Hotel, Washington, D. C., January 25–27, 1961.

Forty-second Summer Meeting, Oklahoma State University, Stillwater, Oklahoma, August 28–31, 1961.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, West Virginia University, Morgantown, May 6, 1961.

ILLINOIS, University of Illinois, Urbana, May 12–13, 1961.

INDIANA

IOWA, Simpson College, Indianola, April 21, 1961.

KANSAS

KENTUCKY, Western Kentucky State College, Bowling Green, Spring, 1961.

LOUISIANA-MISSISSIPPI, Buena Vista Hotel, Biloxi, Mississippi, February 17–18, 1961.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, University of Maryland, College Park, December 3, 1960.

METROPOLITAN NEW YORK

MICHIGAN, Wayne State University, Detroit, March 25, 1961.

MINNESOTA

MISSOURI, University of Missouri, Columbia, Spring, 1961.

NEBRASKA, University of Nebraska, Lincoln, April 15, 1961.

NEW JERSEY

NORTHEASTERN, Wesleyan University, Middletown, Connecticut, November 26, 1960.

NORTHERN CALIFORNIA, San Jose State College, January 14, 1961.

OHIO, Ohio Wesleyan University, Delaware, May 6, 1961.

OKLAHOMA

PACIFIC NORTHWEST, University of Washington, Seattle, June 17, 1961.

PHILADELPHIA, Swarthmore College, Swarthmore, Pennsylvania, November 26, 1960.

ROCKY MOUNTAIN, University of Colorado, Boulder, April, 27–29, 1961.

SOUTHEASTERN, Wofford College, Spartanburg, South Carolina, April 7–8, 1961.

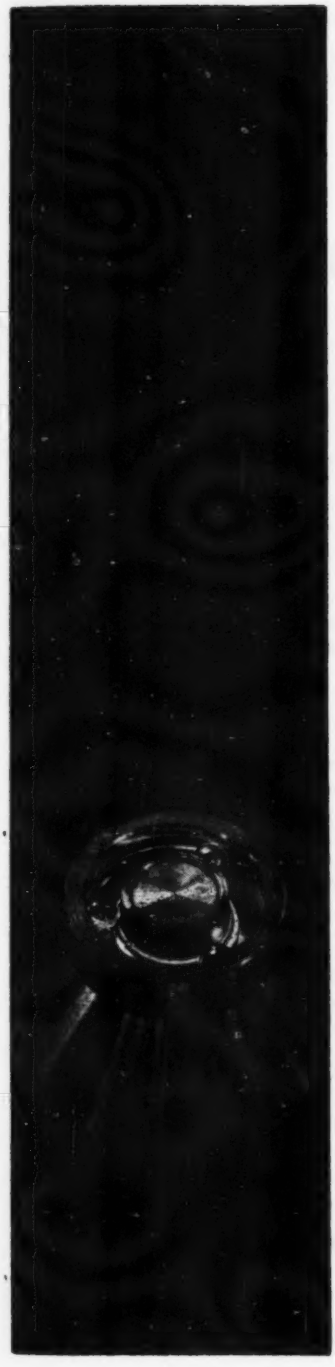
SOUTHERN CALIFORNIA, University of California, Santa Barbara, March 11, 1961.

SOUTHWESTERN, University of Arizona, Tucson, April, 1961.

TEXAS, Stephen F. Austin State College, Nacogdoches, April 14–15, 1961.

UPPER NEW YORK STATE, Harpur College, Binghamton, April 29, 1961.

WISCONSIN, University of Wisconsin, Madison, May 13, 1961.



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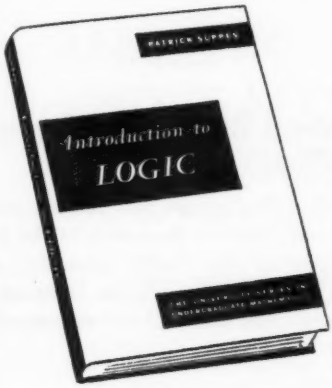

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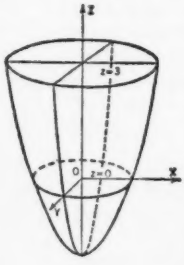

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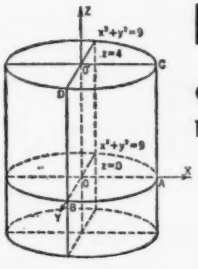

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